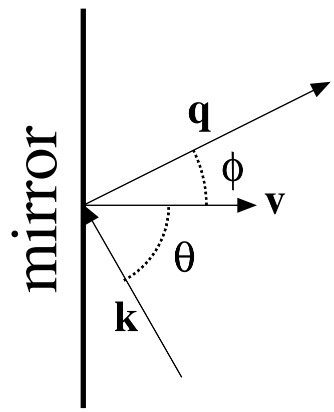
Physics 366: Relativity FINAL EXAM 2026 March

Complete these 4 problems.

1. Consider the problem of light bouncing off a moving mirror ($v = \beta c$). In the displayed frame light has a propagation 4-vector: $\mathbb{k}=(\mathbf{k}, ik)$, where $\mathbf{k} = (-k\cos\theta, k\sin\theta, 0)$. (As required, this is a null 4-vector, with frequency given by $\omega/c = k$.) After reflection the light has a propagation 4-vector: $\mathbb{q}=(\mathbf{q}, iq)$ (with $\omega'/c = q$). You will want to analyze this problem in the rest frame of the mirror (S'), where the usual rules of mirror reflection apply: angle of incidence equals angle of reflection ($\theta'_i = \theta'_r$) with frequency unchanged $\nu'_i = \nu'_r$). So begin by finding: $\mathbb{k}'$, making the bounce by reversing the x' component yielding $\mathbb{q}'$, and finally getting $\mathbb{q}$. As a result show:

$$\cos\phi = \frac{\cos\theta(1+\beta^2)+2\beta}{2\beta\cos\theta+1+\beta^2}$$
$$\sin\phi = \frac{\sin\theta(1-\beta^2)}{2\beta\cos\theta+1+\beta^2}$$



Plugging into the trigonometric identity:

$$\tan\left(\tfrac{1}{2}\phi\right) = \frac{1-\cos\phi}{\sin\phi}$$

should allow you to show a simpler result:

$$\tan\left(\tfrac{1}{2}\phi\right) = \tan\left(\tfrac{1}{2}\theta\right)\,\frac{1-\beta}{1+\beta}$$

2. Consider the distance:

$$ds^2 = da^2 + \sin^2(a)\,db^2$$

with coordinates (a, b). Find the metric tensor g_{ij}. Using your metric tensor, calculate the below non-zero Christoffel symbols (smart students will use the web site).

$$\Gamma^2_{12} = \Gamma^2_{21} = \cot(a) \qquad \Gamma^1_{22} = -\cos(a)\sin(a)$$

Using the above Christoffel symbols, write down the geodesic differential equations for a and b. Recall that if $g_{\mu\nu}$ does not depend on a coordinate (i.e., $g_{\mu\nu,\sigma} = 0$) then the corresponding lowered-index velocity (i.e., dx_σ/ds) is constant. Find this constant of motion for b. Show that the derivative of this constant (which is of course zero) reproduces the b geodesic equation.

3. In PHYS 191, the dot product $\mathbf{A} \cdot \mathbf{B}$ was invariant. Show now that the contraction $A^i B_i$ is invariant under parallel transport i.e.,

$$\delta(A^i B_i) = (\delta A^i)B_i + A^i(\delta B_i) = 0$$

4. The tensor $T^{\mu\nu}$ in the usual $x^\mu = (x, y, z, ct)$ coordinate system has the following values:

$$T^{\mu\nu} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

Find T^μ_ν and use it to produce an invariant. Consider a change of coordinates from rectangular to spherical $x^{\mu'} = (r, \theta, \phi, ct)$ where as usual:

$$\begin{aligned} x &= r\sin\theta\,\cos\phi & \qquad r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\,\sin\phi & \qquad \theta &= \cos^{-1}(z/r) \\ z &= r\cos\theta & \qquad \phi &= \tan^{-1}(y/x) \end{aligned}$$

Find $T^{\mu'\nu'}$, i.e., $T^{\mu\nu}$ in the spherical coordinate system. Given that, as usual, $g_{\mu\nu} = \text{diagonal}(1, 1, 1, -1)$ write down $g_{\mu'\nu'}$. Show $T^{\mu'}{}_{\mu'} = T^{\mu}{}_{\mu}$

This is a very messy problem; please use Mathemtica! I've used `Simplify` and `Assumptions` to get what I wanted. Here is an intermediate step you can use as a check: the coordinate transform matrix re-expressed in spherical coordinates.

$$\frac{\partial x^{\mu'}}{\partial x^\nu} = \begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta & 0 \\ \dfrac{\cos\phi\cos\theta}{r} & \dfrac{\sin\phi\cos\theta}{r} & -\dfrac{\sin\theta}{r} & 0 \\ -\dfrac{\sin\phi\csc\theta}{r} & \dfrac{\cos\phi\csc\theta}{r} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that if the spacial part of $T^{\mu\nu}$ is diagonal, i.e., $T^{ij} = a\overleftrightarrow{\mathbf{1}}$ or $a = b = c$ $T^{i'j'}$ has a simple form:

$$T^{i'j'} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & \dfrac{1}{r^2\sin^2\theta} \end{pmatrix}$$