**Complete 5 of the following 7 problems.**

The $S'$ frame moves with a velocity $v$ down the positive $x$ axis of the $S$ frame. Using the usual parameters: $\beta = v/c$ and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$, the relationship between coordinates in the two frames is given by:

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \qquad \text{and} \qquad \begin{matrix} y' = y \\ z' = z \end{matrix}$$

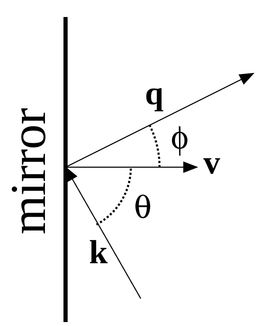
**or:** $\mathbb{X}' = O \cdot \mathbb{X}$ where: $\mathbb{X} = (\mathbf{r}, ict)$

and $O$ is the orthogonal matrix: $\begin{pmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$

1. (a) A really, really, really massive racket (mass: $M \to \infty$) travelling at speed $v$ hits a teeny little ball (mass: $m$) at rest. What is the subsequent velocity of the ball? (Hint: This is a Lorentz transformation problem. Analyze the problem in the easy frame —where the racket is at rest— and transform back to the lab frame. Note that Newton would answer this $2v$.)

   (b) Consider basically the same problem but with light bouncing off a mirrorred racket. Of course, the light could not be initially at rest, rather it has a propagation 4-vector: $\mathbb{k}$=$(\mathbf{k}, ik)$. (As required, this is a null 4-vector, with frequency given by $\omega/c = k$ or $\nu/c = 1/\lambda$.) After reflection the light has a propagation 4-vector: $\mathbb{q}$=$(\mathbf{q}, iq)$ (with $\omega'/c = q$). Once again you will want to analyze this problem in the rest frame of the mirror, where the usual rules of mirror reflection ($\theta_i = \theta_r$ and $\nu_i = \nu_r$) apply. Show:

$$\cos\phi = \frac{\cos\theta(1+\beta^2)+2\beta}{2\beta\cos\theta+1+\beta^2}$$
$$\sin\phi = \frac{\sin\theta(1-\beta^2)}{2\beta\cos\theta+1+\beta^2}$$




   Using the trigonometric identity:

$$\tan\left(\tfrac{1}{2}\phi\right) = \frac{1-\cos\phi}{\sin\phi}$$

   should allow you to show more a usefull result:

$$\tan\left(\tfrac{1}{2}\phi\right) = \tan\left(\tfrac{1}{2}\theta\right)\,\frac{1-\beta}{1+\beta}$$

2. (a) As a neutral pion ($\pi^0$) (rest mass $m$, velocity $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$) flys though the lab, it decays into two (generally different frequency) photons ($\pi^0 \to \gamma\gamma'$). Just using 4-vector algebra, show that the energy of a photon (viewed in the lab frame moving at an angle $\theta$ to the direction of motion of the $\pi^0$) is given by:

$$E = \frac{mc^2}{2\gamma(1-\beta\cos\theta)}$$

Explain why in the rest frame of the $\pi^0$ the two photons must have the same energy: $\frac{1}{2}mc^2$.

(b) Show that a whole parade of photons (perhaps with different frequencies) but all moving in the same direction, produce a total momentum 4-vector that is null. Use this result to show that an interstellar spacecraft 'burning' mass to produce a straight-line photon exhaust will have a mass ratio:

$$\frac{M_{\text{final}}}{M_{\text{initial}}} = \sqrt{\frac{1-\beta}{1+\beta}}$$

where $\beta$ describes the final speed achieved by the spacecraft.

3. Consider the coordinate system $dx^a = (d\chi, d\theta, d\phi)$ with invariant distance:

$$ds^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Note that for small $\chi$ this is like spherical coordinates where:

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Find the non-zero Christoffel symbols.

I claim that with this metric:

$$R_{abcd} = K\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) = K\, g_{a[c}g_{d]b}$$

for some scaler constant $K$ (note that all indices are covariant in this expression). Calculate <u>one</u> non-zero $R_{abcd}$, and then use it in the above expression to determine $K$. Once you've found $K$, calculate the Ricci tensor and the scaler curvature.

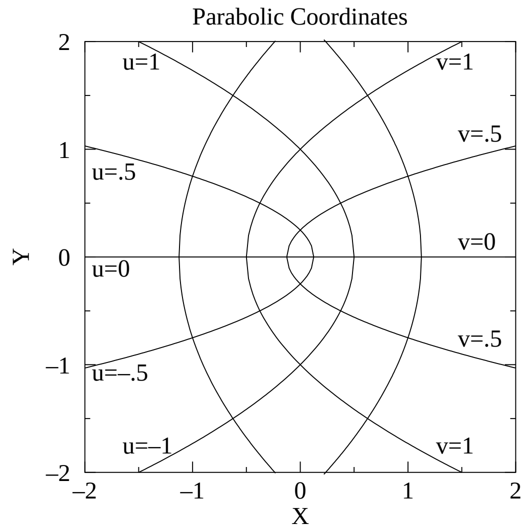
4. Recall our work on uniform acceleration. Equation 8.56 claims that

$$\gamma^*\eta^2 = \text{constant for free fall}$$

Prove this result. Hint: use a combination of *Mathematica* and some hand calculations—this problem is much like homework showing that the coordinate speed of light is $\eta$. (Please turn in hardcopy showing your computer calculations.)

5. Consider the (2d) parabolic coordinate system $(u, v)$:

$$
\begin{aligned}
x &= (u^2 - v^2)/2 \\
y &= uv \\
\\
u &= \frac{y}{\sqrt{r-x}} \\
v &= \sqrt{r-x}
\end{aligned}
$$




where $r = \sqrt{x^2 + y^2}$. Find the vectors: $\mathbf{e}_u$ and $\mathbf{e}_v$. By taking derivatives of these vectors find the Christoffel symbols: $\Gamma^u_{uu}, \Gamma^u_{uv}, \Gamma^u_{vv}, \Gamma^v_{uu}, \Gamma^v_{uv}, \Gamma^v_{vv}$. Use dot products of $\mathbf{e}_u$ and $\mathbf{e}_v$ to find the metric tensor. Take derivatives of the metric tensor to confirm one of the Christoffel symbols determined above.

6. The tensor $T^{\mu\nu}$ in the usual $x^\mu = (x, y, z, ct)$ (flat) coordinate system has the following values:

$$
T^{\mu\nu} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}
$$

Find $T^\mu_\nu$ and use it to produce an invariant. Consider a change of coordinates from rectangular to spherical $x^{\mu'} = (r, \theta, \phi, ct)$ where as usual:

$$
\begin{aligned}
x &= r \sin\theta \, \cos\phi & \qquad r &= \sqrt{x^2 + y^2 + z^2} \\
y &= r \sin\theta \, \sin\phi & \qquad \theta &= \cos^{-1}(z/r) \\
z &= r \cos\theta & \qquad \phi &= \tan^{-1}(y/x)
\end{aligned}
$$

Find $T^{\mu'\nu'}$, i.e., $T^{\mu\nu}$ in the spherical coordinate system.

7. Consider Kirkman's double-door, catch-and-release, rod-in-garage problem. A 10 m long garage has both front and back doors. A rod (rest length 20 m) approaches the (open) front door of the garage at a speed (what speed?) giving $\gamma = 2$. Because the rod appears Lorentz contracted to observers in the garage rest frame ($S$), it should be just possible to —for an instant— close both doors of the garage while the rod is *inside* the garage. However, to an observer in the rest frame ($S'$) of the rod, the garage is contracted and there is no hope of fitting the rod inside the garage. Set up the coordinates systems $S$ and $S'$ so that $x = 0, t = 0, x' = 0, t' = 0$ corresponds to the rear of the rod at front of the garage. Thus at $t = 0$ we can close and quickly open the garage doors and be assured that the rod is inside. Find $S$ and $S'$ coordinates for the event that is the closing (and opening) of the *back* garage door. Describe what is seen in the rest frame of the rod ($S'$). Include in your description: the length of the garage, and show that the front of the garage has time to travel from the event of the closing of the back garage door on the front end of the rod to the event of closing of the front garage door on the rear end of the rod. Perhaps starting with the online blank Minkowski Diagram accurately draw the $S'$ coordinates axies $t' = 0$ and $x' = 0$. On this diagram draw and label the worldline of (A) the garage front, (B) the garage back, (C) the rod rear, and (D) the rod front. Label with (E) the event that is the closing of the front door and label with (F) the event that is the closing of the back garage door.
   If at $t' = 0$, the rod is dipped, so as to leave marks in the ground at the front and rear of the rod, find the garage-rest-frame ($S$) location of the marks. Label the events of making these marks (G) and (H) on your Minkowski Diagram. For the rod-dip, describe the sequence of events as seen in $S$.