

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 - \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha} \quad \int_0^{+\infty} x e^{-\alpha x^2} dx = \frac{1}{2\alpha} \quad \int_0^{\infty} x^n e^{-\alpha x} dx = n!/\alpha^{n+1}$$

$$H\psi = i\hbar\partial_t\psi \quad H\psi = E\psi \quad p = -i\hbar\partial_x \quad [p, x] = -i\hbar$$

$$\partial_t \psi^*(x, t)\psi(x, t) = -\partial_x J \quad \text{where current } J = \frac{\hbar}{2im} (\psi^* \partial_x \psi - \psi \partial_x \psi^*) = \frac{\hbar}{2im} \psi^* \overleftrightarrow{\partial}_x \psi$$

$$\frac{d}{dt} \langle \psi | A | \psi \rangle = \langle \psi | \partial_t A | \psi \rangle + \langle \psi | i[H, A] / \hbar | \psi \rangle$$

$$\sigma_A \sigma_B = \Delta A \Delta B \geq \frac{1}{2} |\langle \psi | i[A, B] | \psi \rangle|$$

Free Particle: $u_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ or $u_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$ where $p = k\hbar$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx} g(k) dk \quad g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$$

Particle-in-a-box with $V(x) = 0$ for $|x| < a$, but $V(x) = \infty$ elsewhere

$$E_n^+ = \frac{(q\hbar)^2}{2m} \quad u_n^+(x) = \frac{1}{\sqrt{a}} \cos(qx) \quad \text{where } q = \frac{(n - \frac{1}{2})\pi}{a} = \frac{(2n - 1)\pi}{2a} = \frac{(\text{odd } N)\pi}{2a}$$

$$E_n^- = \frac{(q\hbar)^2}{2m} \quad u_n^-(x) = \frac{1}{\sqrt{a}} \sin(qx) \quad \text{where } q = \frac{n\pi}{a} = \frac{(2n)\pi}{2a} = \frac{(\text{even } N)\pi}{2a}$$

shifted origin: $E_n = \frac{(q\hbar)^2}{2m} \quad u_n(x) = \sqrt{\frac{2}{L}} \sin(qx) \quad \text{where } q = \frac{n\pi}{L}$

Particle-in-a-box with $V(x) = -V_0$ for $|x| < a$, but $V(x) = 0$ elsewhere

$$z_0^2 \frac{(\hbar/a)^2}{2m} = V_0 \quad -\frac{\kappa^2 \hbar^2}{2m} = E \quad z^2 \frac{(\hbar/a)^2}{2m} = \frac{q^2 \hbar^2}{2m} = E + V_0 = KE \quad \text{where: } z = qa$$

$$E_n^+ = (z^2 - z_0^2) \frac{(\hbar/a)^2}{2m} \quad u_n^+(x) = \begin{cases} \cos(qx) & |x| < a \\ e^{-\kappa|x|} & |x| > a \end{cases} \quad \text{where } \tan z = \sqrt{(z_0/z)^2 - 1} \quad z \approx \frac{(\text{odd } N)\pi}{2}$$

$$E_n^- = (z^2 - z_0^2) \frac{(\hbar/a)^2}{2m} \quad u_n^-(x) = \begin{cases} \sin(qx) & |x| < a \\ \pm e^{-\kappa|x|} & |x| > a \end{cases} \quad \text{where } \cot z = -\sqrt{(z_0/z)^2 - 1} \quad z \approx \frac{(\text{even } N)\pi}{2}$$

reflection: (shifted origin) $\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ A \cos qx + B \sin qx & 0 < x < L \\ Te^{ikx} & x > L \end{cases} \quad \text{where } R = \frac{i(q^2 - k^2) \sin qL}{2qk \cos qL - i(q^2 + k^2) \sin qL}$

Harmonic Oscillator with $V(x) = \frac{1}{2} m\omega^2 x^2 \quad E_n = \hbar\omega (n + \frac{1}{2})$

$$|n\rangle = u_n(x) = N_n H_n(\xi) e^{-\frac{1}{2}\xi^2} \quad \text{where } \xi = \sqrt{\frac{m\omega}{\hbar}} x \quad \text{and } H_n \text{ is an } n^{\text{th}} \text{ degree polynomial}$$

$$a_- = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega} \right) = \frac{1}{\sqrt{2}} (\xi + \partial_\xi) \quad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - i \frac{p}{m\omega} \right) = \frac{1}{\sqrt{2}} (\xi - \partial_\xi)$$

$$[a_-, a_+] = 1 \quad [H, a_-] = -\hbar\omega a_- \quad [H, a_+] = \hbar\omega a_+ \quad H = \hbar\omega \left(\frac{1}{2} + a_+ a_- \right)$$

$$a_- |n\rangle = \sqrt{n} |n-1\rangle \quad a_+ |n\rangle = \sqrt{n+1} |n+1\rangle$$

Delta Function with $V(x) = -\alpha\delta(x) \implies \Delta\psi' = -\frac{2m\alpha}{\hbar^2} \psi(0)$

$$\psi(x) = \sqrt{\kappa} e^{-\kappa|x|} \quad \text{where: } \kappa = m\alpha/\hbar^2 \quad E = -\frac{\kappa^2 \hbar^2}{2m}$$

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & x < 0 \\ Te^{ikx} & x > 0 \end{cases} \quad \text{where } R = \frac{-1}{ik/\kappa + 1}$$