Complex variable - contour integration - residues

$\int_a^b f(z)dz$ - indep of path on complex plane between a & b - like work done by conservative force

$\oint f(z)dz = 0$ - unless encloses singularity (like gauss counts only enclosed point charges not dipoles etc. Here only things like $\frac{1}{z-z_0}$ "poles" count

$\oint f(z)dz = 2\pi \sum$ residues ← coef of pole

↖ only sum over enclosed poles

Greens functions - propagators - delta function source

Recall from E & M: seek solution of $-\nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$

location of δ

then Poisson: $-\nabla^2 V = \rho/\varepsilon_0$ ⇒ $V = \int \frac{\rho(r_0)}{\varepsilon_0} G(r, r_0)\, d^3 r_0$

$G(r, r_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}$

where seeking V

integrate over source location

$$-\nabla^2 \psi + \frac{2mV}{\hbar^2}\psi = k^2\psi$$

seek solution $(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}_0) = \delta^3(\vec{r}-\vec{r}_0)$

$$\psi(r) = \int \frac{2m V(\vec{r}_0)}{\hbar^2}\psi(r_0)\, G(\vec{r}, \vec{r}_0)\, d^3 r_0 + \text{"free" solution}$$

ie homogeneous solution.

$(\nabla^2+k^2)\,G = \delta(\vec r-\vec r_0)$ ← shift origin to $\vec r_0$

Fourier: $G(\vec r)=\frac{1}{(2\pi)^3}\int e^{i\vec s\cdot\vec r}\, g(s)\, d^3s$

$$\delta(r)=\frac{1}{(2\pi)^3}\int e^{i\vec s\cdot\vec r}\, d^3s$$

$(k^2-s^2)\,g(s)=1$

$src \qquad s^2\,ds\,dc\,d\phi \nearrow 2\pi$

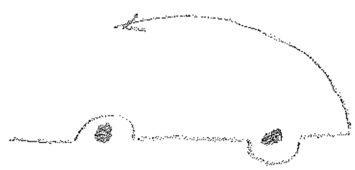
$$G(r)=\frac{1}{(2\pi)^3}\int e^{i\vec s\cdot\vec r}\,\frac{1}{k^2-s^2}\,d^3s$$

$$=\frac{1}{(2\pi)^2}\int \frac{1}{isr}\,\frac{e^{isrc}\Big|_{-1}^{1}}{k^2-s^2}\, s^2\,ds$$

$$=\frac{2}{(2\pi)^2}\,\frac{1}{r}\int \frac{\sin(sr)\;s\,ds}{k^2-s^2}$$ ← even

$$=\frac{1}{(2\pi)^2}\,\frac{1}{r}\int_{-\infty}^{\infty}\frac{e^{isr}-e^{-isr}}{(k^2-s^2)\,2i}\,s\,ds$$

↑ poles @ $s=\pm k$



$$\frac{1}{-2i(s^2-k^2)}=\underbrace{\frac{1}{-2i(s+k)}}_{\frac{1}{-4ik}}\;\frac{1}{(s-k)}$$

$$2\pi i\,\frac{k}{-4ik}\,e^{ikr}$$

$$\frac{1}{-2i(s^2-k^2)}=\underbrace{\frac{1}{-2i(s-k)}}_{\frac{1}{+4ik}}\;\frac{1}{s+k}$$

$$2\pi i\,\frac{-k}{4ik}\,e^{ikr}$$

$$\Rightarrow \frac{1}{(2\pi)^2}\,\frac{1}{r}\left(-\pi e^{ikr}\right)$$

$$=\frac{-1}{4\pi r}\,e^{ikr}\longrightarrow \frac{-1}{4\pi|\vec r-\vec r_0|}\,e^{ik|\vec r-\vec r_0|}$$

$$|\vec{r}-\vec{r}_0|^2 = r^2 - 2\vec{r}\cdot\vec{r}_0 + r_0^2 \approx r^2\left(1 - 2\frac{\hat{r}\cdot\vec{r}_0}{r}\right)$$

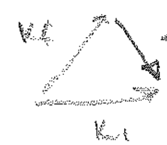
$$|r-r_0| \approx r\left(1 - \frac{\hat{r}\cdot\vec{r}_0}{r}\right)$$

$$\frac{e^{ik|r-r_0|}}{|r-r_0|} \approx \frac{1}{r}\, e^{ikr}\, e^{-i\overbrace{k\,\hat{r}}^{\vec{K}_f}\cdot r_0}$$

$$\psi = \underset{\to\, e^{i\vec{K}_i\cdot\vec{r}}}{\psi_0} - \frac{m}{2\pi\hbar^2}\,\frac{e^{ikr}}{r}\int e^{-i\vec{K}_f\cdot r_0}\, V(r_0)\, \underset{\to\, e^{i\vec{K}_i\cdot\vec{r}}}{\psi(r_0)}\, d^3r_0$$

$$e^{i\underbrace{(K_i-K_f)}_{\vec{q}}\cdot r_0}\, V(r_0)$$

$\vec{q}$ "momentum transfer"



$\leftarrow 2k\sin\left(\frac{\theta}{2}\right)$

$$f = \frac{-m}{2\pi\hbar^2}\; FT\,(2k\sin\theta)$$

small k: $\quad V\cdot\frac{4}{3}\pi R^3 \quad$ (wrong)