Example 2 - SHO in 2d

only possible rotation in 2d

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2$$ relates to L_z $\to e^{im\phi} \Rightarrow -m^2$

$$\Big(\underbrace{-\frac{\hbar^2}{2m}}_{A}\nabla^2 + \underbrace{\frac{1}{2}m\omega^2}_{B} r^2\Big) R e^{im\phi} = E\, R e^{im\phi}$$

A units $E\cdot L^2$ B units $\frac{E}{L^2}$

$$\ell = (A/B)^{1/4} = \left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$e = (A\cdot B)^{1/2} = \frac{1}{2}\hbar\omega$$

dimensionless: $x' = \frac{x}{\ell}$ $E' = E/e$ drop primes

$$(-\nabla^2 + r^2) R e^{im\phi} = E\, R e^{im\phi}$$

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + r^2\right) R = E R$$ ← note: E depends on $|m|$

or $$\left[-(\sqrt{r}R)''\right] + \left[\frac{m^2 - 1/4}{r^2} + r^2\right]\sqrt{r}R = E\sqrt{r}R$$

① seek behavior as $r \to \infty$: $(\sqrt{r}R)'' \approx r^2 \sqrt{r} R$

$\approx e^{-\frac{1}{2}r^2}$

② seek behavior as $r \to 0$

$$R \sim r^q \to -q(q-1)r^{q-2} - q\, r^{q-2} + m^2 r^{q-2} \approx 0$$

$$\to q^2 = m^2$$

③ Try solution: $R = r^{|m|} e^{-\frac{1}{2}r^2} G$; now drop $|m|$

$$R' = \left(\frac{m}{r} - r\right) R + r^m e^{-\frac{1}{2}r^2} G'$$

$$rR' = (m - r^2) R + r^{m+1} e^{-\frac{1}{2}r^2} G'$$

$$(rR')' = -2rR + (m - r^2)\left[\left(\frac{m}{r} - r\right)R + r^m e^{-\frac{1}{2}r^2} G'\right]$$

$$+ \left(\frac{m+1}{r} - r\right) G' + r^{m+1} e^{-\frac{1}{2}r^2} G''$$

$$-G'' - \left[\frac{2m+1}{r} - 2r\right] G' + \left[2(m+1) - E\right] G = 0$$

$$-(k+2)(k+1)\, a_{k+2} - (2m+1)(k+2)\, a_{k+2} + 2k\, a_k + [2(m+1) - E]\, a_k = 0$$

$$a_{k+2} = \frac{(-E + 2m + 2 + 2k)}{(k+2)(k+2m+2)}\, a_k$$ ← $k = \text{even} = 2i$

$a_{i+1} = (\ )\, a_i$

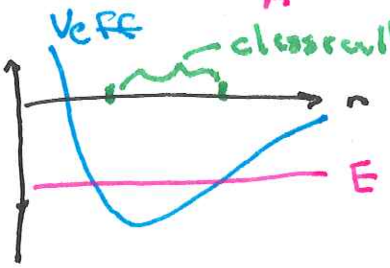
$$a_{i+1} = \frac{\left(-\overbrace{(\tfrac{1}{2}E - m - 1)}^{\to 2n}/2 + i\right)}{(1+i)(m+1+i)}\, a_i \qquad \longrightarrow \quad 2n + m + 1 = \tfrac{1}{2}E$$

$$\downarrow$$

$$E = 2(n + |m| + 1)$$

$$= {}_1F_1\left(\frac{-n}{m+1};\, r^2\right) \longrightarrow L_n^m(r^2)$$

H-atom: $\left(-\underbrace{\frac{\hbar^2}{2m}}_{A}\left(\partial_r^2+\frac{2}{r}\partial_r\right) + \frac{\hbar^2 l(l+1)}{2mr^2} - \overset{\alpha}{\frac{Ze^2}{4\pi\epsilon_0 r}}\right)R = E\,R$



① units:

$A = \frac{\hbar^2}{2m} = E\cdot L^2 \qquad \alpha = E\cdot L$

length unit $l = \frac{2A}{\alpha}$ "Bohr radius"

energy unit $e = \frac{\alpha}{l} = \frac{1}{2}\frac{\alpha^2}{A}$ "hartree"

$$\frac{\hbar^2}{2m}\nabla^2 \rightarrow \frac{\hbar^2}{2ml^2}\nabla'^2 = \frac{A}{(2A/\alpha)^2}\nabla'^2 = \frac{1}{2}\frac{\alpha^2}{2A}\nabla'^2$$

$$\frac{Ze^2}{4\pi\epsilon_0 r} = \frac{Ze^2}{4\pi\epsilon_0 l}\frac{1}{r'} = \frac{\alpha}{(2A/\alpha)}\frac{1}{r'} = \frac{\alpha^2}{2A}\frac{1}{r'}$$

$$\left(-\frac{1}{2}\nabla'^2 - \frac{1}{r'}\right)\psi = E'\psi \qquad \psi = R\,Y_l^m$$

$$\left(-\left(\partial_r^2+\frac{2}{r}\partial_r\right) + \frac{l(l+1)}{r^2} - \frac{2}{r}\right)R = 2\underset{\hookrightarrow -|E|}{E}R \quad \text{drop primes}$$

② Behavior as $r\to\infty$ $\qquad -\partial_r^2 R = -2|E|R$

$$R \propto e^{-\sqrt{2|E|}\,r} \equiv e^{-\frac{1}{2}\rho} \qquad \rho \equiv \sqrt{8|E|}\,r$$

divide DiffEq by $\frac{1}{8|E|}$

$$\left(-\left(\partial_\rho^2+\frac{2}{\rho}\partial_\rho\right) + \frac{l(l+1)}{\rho^2} - \overset{\text{define as }\lambda}{\frac{2}{\sqrt{8|E|}}}\frac{1}{\rho}\right)R = -\frac{1}{4}R$$

$$0 = \Big[\underbrace{\left(\partial_\rho^2+\frac{2}{\rho}\partial_\rho\right) - \frac{l(l+1)}{\rho^2}}_{\text{largest part as }\rho\to 0} + \frac{\lambda}{\rho} - \frac{1}{4}\Big]R$$

③ Behavior as $\rho\to 0$

Try ρ^q: $\quad 0 = \Big[\underbrace{q(q-1)+2q}_{q(q+1)} - l(l+1)\Big]\rho^{q-2}$

solutions, $q = l$ or $\underbrace{-(l+1)}_{\text{Not normalizable}}$

(4) Factor out large/small behavior: $R = \rho^{\ell} e^{-\frac{1}{2}\rho} H$

$$R' = \left(\frac{\ell}{\rho} - \frac{1}{2}\right) R + \rho^{\ell} e^{-\frac{1}{2}\rho} H'$$

$$\rho^2 R' = \left(\ell\rho - \frac{1}{2}\rho^2\right) R + \rho^{\ell+2} e^{-\frac{1}{2}\rho} H'$$

$$(\rho^2 R')' = (\ell - \rho) R + \left(\ell\rho - \frac{1}{2}\rho^2\right)\left[\left(\frac{\ell}{\rho} - \frac{1}{2}\right) R + \rho^{\ell} e^{-\frac{1}{2}\rho} H'\right]$$
$$+ \left(\frac{\ell+2}{\rho} - \frac{1}{2}\right) \rho^{\ell+2} e^{-\frac{1}{2}\rho} H' + \rho^{\ell+2} e^{-\frac{1}{2}\rho} H''$$

$$\frac{\frac{1}{\rho^2}(\rho^2 R')'}{\rho^{\ell} e^{-\frac{1}{2}\rho}} = \left(\frac{\ell}{\rho^2} - \frac{1}{\rho}\right) H + \left(\frac{\ell}{\rho} - \frac{1}{2}\right)\left[\left(\frac{\ell}{\rho} - \frac{1}{2}\right) H + H'\right]$$
$$+ \left(\frac{\ell+2}{\rho} - \frac{1}{2}\right) H' + H''$$

$$0 = \boxed{} - \frac{\ell(\ell+1)}{\rho^2} H + \frac{\lambda}{\rho} H - \frac{1}{4} H$$

$$0 = H'' + \left(\frac{2\ell+2}{\rho} - 1\right) H' + \frac{\lambda - \ell - 1}{\rho} H$$

(5) Try poly solution for $H = \sum a_k \rho^k$

$$0 = \Big[\underbrace{k(k-1) + k(2\ell+2)}_{k(2\ell+k+1)}\Big] a_k \rho^{k-2} + (k + \ell + 1 - \lambda) a_k \rho^{k-1}$$

shift $k \to k+1$

$$a_{k+1} = \frac{(\overbrace{\ell+1-\lambda} + k)}{(k+1)(2\ell+2+k)} a_k$$

require $= -n_r$

so $\lambda = n_r + \ell + 1$

$$\Rightarrow \quad H = {}_1F_1\left(\begin{matrix} -n_r \\ 2\ell+2 \end{matrix}; \rho\right) \leftarrow \text{Laguerre } L^{2\ell+1}_{n_r}$$

$$\lambda = \frac{2}{\sqrt{8|E|}} \Rightarrow |E| = \frac{1}{2\lambda^2} = \frac{1}{2(\underbrace{n_r + \ell + 1}_{\text{define as } n})}$$