classical mechanics = mechanics of avgs

$$\frac{d}{dt}\langle Q\rangle = \langle \partial_t \psi | Q | \psi\rangle + \langle \psi | Q | \partial_t \psi\rangle + \langle \psi | \underbrace{\partial_t Q}_{\text{often zero}} | \psi\rangle$$

$$\langle \partial_t \psi| \leftarrow \left(\tfrac{1}{i\hbar} H\psi\right)^*, \qquad |\partial_t\psi\rangle \leftarrow \tfrac{1}{i\hbar} H\psi$$

$$= \frac{1}{-i\hbar}\left\{ \langle H\psi | Q | \psi\rangle - \langle \psi | Q | H\psi\rangle \right\} + \langle \psi | \partial_t Q | \psi\rangle$$

$$= \frac{i}{\hbar}\langle \psi | [H, Q] | \psi\rangle + \langle \psi | \partial_t Q | \psi\rangle$$

in classical mechanics we have an identical looking relation

$$\frac{d}{dt} Q = [Q, H] + \frac{\partial Q}{\partial t}$$ "Poisson Bracket"

so in both QM & classical mechanics a time independent quantity will be a constant of the motion if it commutes with the Hamiltonian.

In QM we will be concerned with quantities that commute with H as then if $H\psi = E\psi$ then $H(Q\psi) = E(Q\psi)$ ie we can generate degenerate eigenfunctions using Q

Pf: $HQ = QH$ if $[Q, H] = 0$ & $HQ\psi = QH\psi = E\,(Q\psi)$

Recall H atom from Modern: eigenenergy $E_n = \frac{-13.6\,eV}{n^2}$ but for every n there are n^2 degenerate states that have the same energy but different l & m

Eg for $n=3$ we have 5 d ($l=2$) states, 3 p ($l=1$), and 1 s ($l=0$)

Remark: "mechanics of averages" — one source of difference is things like $\langle V\rangle \neq V(\langle x\rangle)$

such things will be nearly true if σ_x small compared to range over which V varies

Generalize uncertainty: $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}[A,B]\right)^2$

Note: For Hermitian A, B: $[A,B]^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[A,B]$

so we expect $[A,B]$ to be pure imaginary hence the $\frac{1}{i}$ in above $\rightarrow$ it will cancel with an i from $[A,B]$ eg $[x,p] = i\hbar$; $[L_x, L_y] = i\hbar L_z$

Pf: wolog we assume $\langle A \rangle$ & $\langle B \rangle = 0$ as if $\langle A \rangle \neq 0$ then $A' = A - \langle A \rangle$ will have $\langle A' \rangle = 0$ and A' will have the same stdev as A & same commutator as A

So $\sigma_A^2 = \langle \psi | A^2 | \psi \rangle = \langle A\psi | \underbrace{A\psi}_{f} \rangle$

$\sigma_B^2 = \langle \psi | B^2 | \psi \rangle = \langle B\psi | \underbrace{B\psi}_{g} \rangle$

CBS: $\langle f|f \rangle \langle g|g \rangle \geq |\langle f|g \rangle|^2 \geq \left(\text{Im} \langle f|g \rangle\right)^2$

$\text{Im} \langle f|g \rangle = \frac{1}{2i}\left(\langle f|g \rangle - \langle g|f \rangle\right) = \frac{1}{2i}\left(\langle \psi|AB|\psi \rangle - \langle \psi|BA|\psi \rangle\right)$

$= \frac{1}{2i}\langle [A,B] \rangle$

Note: t has a different status cf. x
there is no time operator; t is independent variable

How get $\Delta t \, \Delta E \geq \frac{\hbar}{2}$?

Let $\Delta t \equiv$ time it takes Q to change by σ_Q

$\Delta t \, \Delta E = \frac{\sigma_Q \sigma_H}{\left|\frac{d\langle Q \rangle}{dt}\right|} = \frac{\sigma_Q \sigma_H}{\left|\frac{i}{\hbar}\langle [H,Q] \rangle\right|} = \frac{\hbar}{2}$ $\quad \sigma_Q \sigma_H \geq \left|\frac{1}{2i}[Q,H]\right|$