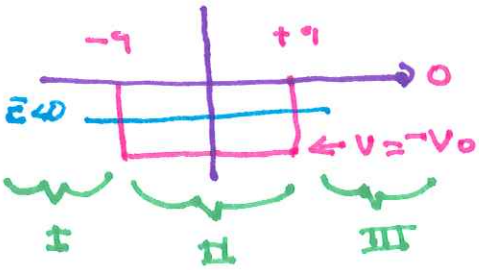
Example 5: finite square well – graphical method

upshot: a handful of bound states ($E<0$, even/odd, normalizable)

∞ scattering states ($E>0$, unnormalizable – like e^{ikx})



$$\psi'' = \underbrace{\frac{-2m(E-V)}{\hbar^2}}\,\psi$$

→ in region I & III : $V=0$; $E<0$

so this constant is >0; $+\kappa^2$ kappa

→ in region II : $V=-V_0$; $E+V_0>0$

so this constant is <0; $-k^2$

in region II $\quad \psi = N\begin{cases}\sin kx \\ \cos kx\end{cases}$ depending on even/odd solution

in region III $\quad \psi = \tilde{N} e^{-\kappa x}$

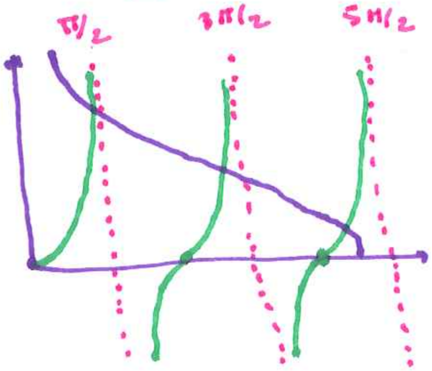
@ $x=+a$ $\quad\psi$ continuous & ψ' continuous; equate $\frac{\psi'}{\psi}$ "log derivative"

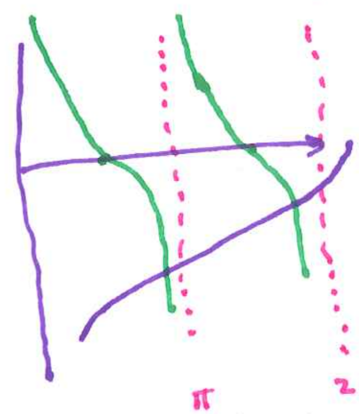
$$\begin{matrix}\text{odd} \\ \text{even}\end{matrix}\left[\begin{matrix} k\cos(ka)/\sin(ka) \\ -k\sin(ka)/\cos(ka)\end{matrix}\right. = -\kappa \leftarrow \text{kappa}$$

note: $\quad k^2 = \frac{2m(V_0+E)}{\hbar^2} = \frac{2mV_0}{\hbar^2} - \kappa^2$

$$z^2 \equiv k^2a^2 = \frac{2mV_0a^2}{\hbar^2} - \kappa^2a^2 \equiv V_0' - \kappa^2a^2$$

$$\left[\begin{matrix} z\cot z = -\kappa a = -\sqrt{V_0'-z^2} \\ z\tan z = +\sqrt{V_0'-z^2}\end{matrix}\right. \longleftrightarrow \begin{matrix}\cot z = -\sqrt{\frac{V_0}{z^2}-1} \\ \tan z = +\sqrt{\frac{V_0}{z^2}-1}\end{matrix}$$





For large V_0', intersections $\sim \frac{(\text{odd})\pi}{2}$

$$\Rightarrow \underbrace{E-V}_{KE} = V_0+E = \frac{\hbar^2k^2}{2m} = \frac{\hbar^2}{2ma^2}z^2$$

$$\approx \frac{\hbar^2\pi^2}{2m(2a)^2}(\text{odd})^2$$

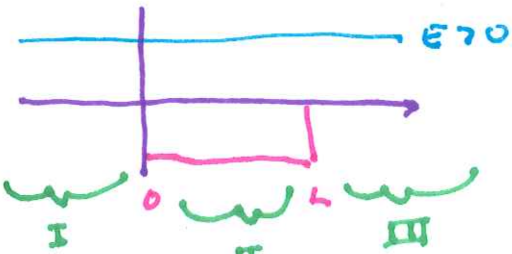
For large V_0' intersections $\sim \frac{(\text{even})\pi}{2} \Rightarrow$

$$E-V \approx \frac{\hbar^2\pi^2}{2m(2a)^2}(\text{even})^2$$

Final state will be close to $\frac{(\text{even})\pi}{2}$

In general must be between

if $V_0 \to 0$ $\exists$ one solution



$$\psi'' = \underbrace{\frac{-2m(E-V)}{\hbar^2}}\,\psi$$

→ in regions I & III $V=0$, $E>0$
so this constant <0 ; $-k^2$

→ in region II $V=-V_0$, $E>0$
so this constant <0 ; $-g^2$

Note: $g^2 = \frac{2m(E+V_0)}{\hbar^2} = k^2 + \frac{2mV_0}{\hbar^2}$

$$g^2L^2 = k^2L^2 + V_0'$$

$$V_0' = \frac{V_0}{\left\{\frac{\hbar^2}{2mL^2}\right.}$$

different energy unit

I: $\psi = 1e^{ikx} + Re^{-ikx}$

II: $\psi = A\cos gx + B\sin gx$

III $\psi = Te^{ikx}$

match @ $x=0$

ψ: $1+R = A$

ψ': $ik(1-R) = Bg$

$\hookrightarrow$ $1-R = \frac{-ig}{k}B$

add $2 = A - \frac{ig}{k}B$

match @ $x=L$

$A\cos gL + B\sin gL = Te^{ikL}$

$-Ag\sin gL + Bg\cos gL = ikTe^{ikL}$

$\hookrightarrow$ $\frac{ig}{k}(A\sin gL - B\cos gL) = Te^{ikL}$

$\frac{ig}{k}(A\sin gL - B\cos gL) =$
$A\cos gL + B\sin gL$

$$\Rightarrow R = A-1 = \frac{i(g^2-k^2)\sin gL}{2gk\cos(gL) - i(g^2+k^2)\sin gL}$$

Note as $k\to 0$ $R\to -1$
$gL \to \sqrt{V_0'}$

→ if $gL = n\pi$; $R=0$

as $k\to\infty$; $g^2-k^2 = \frac{2mV_0}{\hbar} = \text{constant}$

but denominator $\propto k^2 \Rightarrow R\to 0$