11. (20 points) Consider scattering from an attractive, spherical square-well of radius a. The following two pages contain plots calculated for the potential $V_0 = -30\hbar^2/2ma^2$ (i.e., dimensionless potential $U(r) = -U_0 = -30$ for $r < a$ and $U(r) = 0$ for $r > a$) for various values of the dimensionless momentum: $k = pa/\hbar$. The dimensionless scattering amplitude, f, is the usual scattering amplitude divided by a; The dimensionless total cross-section, σ, is the usual total cross-section divided by a^2. The partial wave phase shifts δ_l have been calculated as functions of k for various values of l (i.e., δ for $l = 0$ is displayed as the red curve, etc.). On the second page the dimensionless differential cross-section ($|f|^2$ as a function of the scattering angle θ) is displayed for an exact (partial wave, in red) calculation and a Born approximation calculation (in blue). Sometimes just the small-angle scattering in the "forward" hemisphere is displayed as a semi-log plot; For $k = 20$ and $k = 40$ normal plots for large scattering angles in the "backward" hemisphere are displayed. Needless to say there is a great deal of information available in these plots!

(a) Define *scattering amplitude* (f) and phase shifts (δ_l).

(b) In the Born approximation the dimensionless momentum transfer $\mathbf{q}$ plays an important role. Define $\mathbf{q}$ and show that:

$$q = 2k\ \sin(\theta/2)$$

FYI: the Born approximation applied to this potential results in:

$$f = \frac{U_0}{q^2}\left(\frac{\sin q}{q} - \cos q\right)$$

(c) The plots show that the Born approximation of $|f|^2$ always seems be 100 at $\theta = 0$ and that there is a small-angle zero in $|f|^2$ at about: $\theta \approx 4.5/k$. Prove these results and using your proofs report a better approximation than the above "4.5". Remark: $|f|^2$ is at its largest between $\theta = 0$ and $\theta \approx 4.5/k$; This region is known as the main diffractive peak.

(d) On the left side of the first page of plots there are two striking features: that the scattering is nearly isotropic for $k \ll 1$ (e.g., at $k = 0.5$) and that, near $k = 1.4656$, σ is unusually large. Explain both of these observations in terms of the the plotted behavior of $\delta_l(k)$.

(e) At $k \approx 1.4656$ $|f|^2$ has gone from nearly isotropic to having three approximate zeros. Explain why three approximate zeros are expected, given the plotted behavior of $\delta_l(k)$.

(f) Define *form factor*. Explain why in, for example, electrons scattering off gold nuclei, you might expect the form factor to resemble the Born approximation for this square-well as reported in (b). for f.