Recall the two-particle CM coordinate transformation:

$$\begin{aligned}
\vec{\mathbf{R}} &= \frac{m_1}{M}\,\vec{\mathbf{r}}_1 + \frac{m_2}{M}\,\vec{\mathbf{r}}_2 & \qquad \vec{\mathbf{r}}_1 &= \vec{\mathbf{R}} + \frac{m_2}{M}\,\vec{\mathbf{r}} \\
\vec{\mathbf{r}} &= \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 & \qquad \vec{\mathbf{r}}_2 &= \vec{\mathbf{R}} - \frac{m_1}{M}\,\vec{\mathbf{r}}
\end{aligned} \tag{1}$$

Consider a two particle state where each particle has definite momentum: $\vec{\mathbf{p}}_i = \hbar \vec{\mathbf{k}}_i$ for $i = 1, 2$:

$$\psi = e^{i(\vec{\mathbf{k}}_1\cdot\vec{\mathbf{r}}_1 + \vec{\mathbf{k}}_2\cdot\vec{\mathbf{r}}_2)} \tag{2}$$

Rewrite this wavefunction using the CM coordinates $\vec{\mathbf{R}}$, $\vec{\mathbf{r}}$. Show that the result can be interpreted as total momentum $\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2$, and relative momentum $\vec{\mathbf{p}} = \mu(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2)$. For the case of identical particles ($m_1 = m_2 = m = M/2$), write down a sum of wavefunctions like (2) that is symmetric under particle exchange, convert the result to the CM coordinates, and simplify to produce a result like:

$$\psi = e^{i\vec{\mathbf{K}}\cdot\vec{\mathbf{R}}}\ \cos(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})$$

Show that antisymmetric under exchange results in something like:

$$\psi = e^{i\vec{\mathbf{K}}\cdot\vec{\mathbf{R}}}\ \sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})$$

Consider two identical particles ($m_1 = m_2 = m = M/2$) that together experience a one dimensional simple harmonic potential:

$$V = \frac{1}{2}\,m\omega^2\left(x_1^2 + x_2^2\right)$$

The Hamiltonian can be expressed as the sum of two identical SHOs:

$$H = \left\{\frac{1}{2m}\,p_1^2 + \frac{1}{2}m\omega^2\,x_1^2\right\} + \left\{\frac{1}{2m}\,p_2^2 + \frac{1}{2}m\omega^2\,x_2^2\right\}$$

The product wavefunction:

$$\psi = |n_1\rangle\,|n_2\rangle$$

can then be seen to be an eigenfunction with eigenenergy that is the sum of the individual SHO engenenergies:

$$E = \hbar\omega\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega\left(n_2 + \tfrac{1}{2}\right)$$

Rewrite this Hamiltonian in terms of the CM coordinates (X, x) and show that the result is still the sum of SHOs, with eigenenergies:

$$E = \hbar\omega\left(N + \tfrac{1}{2}\right) + \hbar\omega\left(n + \tfrac{1}{2}\right)$$

Note that particle exchange $x_1 \leftrightarrow x_2$ becomes: $x \leftrightarrow -x$ and $X \leftrightarrow X$ If the particles are fermions (i.e., if this wavefunction must be antisymmetric [a.k.a. odd] under particle exchange), what values of N, n are allowed? List the N, n for the four lowest energy antisymmetric states. If the particles are bosons (i.e., if the wavefunction must be symmetric [a.k.a. even] under particle exchange), what values of N, n are allowed? List the N, n for the four lowest energy symmetric states.