341 - class 9 - old exam #1, 3-12, HW#1

1(a) General solution to azimuthally symmetric Laplace Eq

$$\phi = \sum \left(A_n r^n + \frac{C_n}{r^{n+1}} \right) P_n(\cos\theta)$$

singular @ r=0

singular @ r=∞ - requires finite E @ r=∞

(b) $\phi_{in}(R) = \sum A_n R^n P_n(\cos\theta) = \phi_{out}(R)$

$= \sum \frac{C_n}{R^{n+1}} P_n(\cos\theta)$

The coef of P_n must match rhs/lhs because

i) The ~~....~~ P_n are unit vectors; same vector ⇒ same coef

ii) More detail: mult both sides by $P_m(\cos\theta)$ & $\int dc$

orthogonality → infinite sum reduced to one term

$A_m R^m = \frac{C_m}{R^{m+1}}$ // we call this game "Fourier Trick"

(c) Given $V(c) = c^2$ & $\int_{-1}^{1} V(c) P_n(c)\, dc = \begin{cases} 2/3 & n = 0 \\ 0 & 1 \\ 4/15 & 2 \\ 0 & \text{rest} \end{cases}$

$V(c) = \phi_{in}(r=R, c) = \sum A_n R^n P_n(c)$

mult by P_m; integrate ⇒ $\int_{-1}^{1} V(c) P_m(c)\, dc = \sum A_n R^n \underbrace{\int_{-1}^{1} P_m(c) P_n(c)\, dc}_{\text{only } m=n \text{ non zero}}$

m	
0	2/3
1	0
2	4/15
3	0

$= A_m R^m \frac{2}{2m+1}$

↳ $m=0$: $\frac{2}{3} = A_0 2 \rightarrow A_0 = \frac{1}{3}$

$m=2$: $\frac{4}{15} = A_2 R^2 \frac{2}{5} \rightarrow A_2 = \frac{2}{3R^2}$

$\phi_{in} = A_0 P_0 + A_2 r^2 P_2$

$= \frac{1}{3} P_0 + \frac{2}{3} \left(\frac{r}{R}\right)^2 P_2$

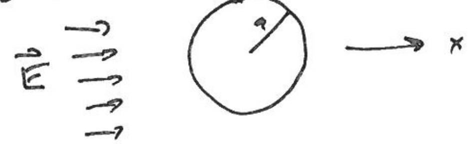
$C_0 = A_0 R = \frac{1}{3} R$

$C_2 = A_2 R^5 = \frac{2}{3R^2} R^5 = \frac{2R^3}{3}$

$\phi_{out} = \frac{A_0}{r} P_0 + \frac{A_2}{r^3} P_2$

$= \frac{1}{3}\left(\frac{R}{r}\right) P_0 + \frac{2}{3}\left(\frac{R}{r}\right)^3 P_2$

3-12



generally for $r \to \infty$ we cannot have terms like $A_n r^n (\cos n\phi$ & $\sin n\phi)$ but here there is an $\vec{E}$ @ $r = \infty \to \phi = -E \underbrace{r\cos\phi}_{x}$

so:

$$\phi = -Er\cos\theta + \sum C_n r^{-n} (\cos n\phi \text{ \& } \sin n\phi)$$

rule out by symmetry - must be even in ϕ

also could have "$n=0$" $\log(r)$ But Gauss says this requires net charge on the cylinder
Rule out

Also - a constant adds nothing

SO: $\phi = -Er\cos\theta + \sum C_n r^{-n} \cos n\phi$

require $\phi(r=R) = \text{const} = V$

$$V = \phi(r=R) = -ER\cos\theta + \sum C_n R^{-n} \cos n\phi$$

By Fouriers Trick the coef of $\cos(m\phi)$ must be the same rhs/lhs since $V = \text{const}$ that coef lhs $= 0$

SO: $-ER + \dfrac{C_1}{R} = 0$ & $C_n = 0$ $n > 1$

$$C_1 = ER^2$$

$$\phi = -Er\cos\theta + \frac{ER^2}{r}\cos\phi$$

Homework 1: we have an <u>even</u> voltage distribution which is $+1$ for $c > c_b$ & -1 for $c < c_b$ (and mirror for bottom hemi)

seek $A_4 \propto \int_{-1}^{1} V(c) P_4(c)\, dc = 2\left\{ \int_0^{c_b} (-1) P_4\, dc + \int_{c_b}^{1} (+1) P_4\, dc \right\}$

$A_4 = 0$

$P_4 = 4^{TH}$ deg poly so $\int P_4 = 5^{TH}$ deg poly
by hand root finding is hard

F[c_] = Integrate[(-1) LegendreP[x, 4], {x, 0, c}]
+ Integrate[(+1) LegendreP[x, 4], {x, c, 1}]

see attached code

```
f[c_,n_]=2(-Integrate[LegendreP[n,x],{x,0,c}]+Integrate[LegendreP[n,x],{x,c,1}] )
FindRoot[f[c,4],{c,.6}]

result: {c -> 0.654654}

g[n_]=f[c,n] /. %
a=Table[N[g[n] (2 n+1)/2],{n,0,10,2}]
phi[r_,c_]=Sum[a[[i]] r^(2 i-2) LegendreP[2 i-2,c],{i,1,6}]

ContourPlot[phi[Sqrt[x^2+z^2],z/Sqrt[x^2+z^2]], {x,-.9,.9},{z,-.9,.9},
ContourShading ->False,Contours -> Table[.05*i +phi[0,0],{i,-10,22}],
RegionFunction ->Function[{x, z}, x^2+z^2 <.9^2], PlotRangePadding->None]

phi2[r_,c_]=Sum[a[[i]] r^(2 i-2) LegendreP[2 i-2,c],{i,1,2}]

Plot[{phi[r,1],phi[r,0],phi2[r,1],phi2[r,0]},{r,0,.9},
PlotStyle -> {Directive[Black],Directive[Black],Directive[Red],Directive[Red]}]
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