Remarks: In dealing with spherical coordinates in general and with Legendre polynomials in particular it is convenient to make the substitution $c = \cos\theta$. For example, this allows use of the following simplification of the orthogonality relationship:

$$\int_0^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2n+1}\,\delta_{nm} \Longrightarrow \int_{-1}^{+1} P_n(c)P_m(c)\, dc = \frac{2}{2n+1}\,\delta_{nm} \quad (1)$$

Since $\theta = \pi/2$ (the equator) corresponds to $c = 0$, symmetries that correspond to reflection in the equatorial plane correspond to $c \to -c$ symmetry. So the statement

$$P_n(-c) = (-1)^n P_n(c) \quad (2)$$

reports that the n-even P_n have even reflection symmetry whereas the n-odd P_n have odd reflection symmetry. Finally note that since $\theta = 0$ and π corresponds to $c = \pm 1$, the statements $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ report the behavior of P_n along the positive and negative z axes respectively.

As shown in the text, we can write an arbitrary azimuthally-symmetric solution to Laplace's equation in spherical coordinates as:

$$\phi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{C_n}{r^{n+1}}\right)P_n(\cos\theta) \quad (3)$$

or equivalently

$$\phi(r,c) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{C_n}{r^{n+1}}\right)P_n(c) \quad (4)$$

Example 1: Consider the problem of finding ϕ inside a sphere (of radius R) where the voltage on the surface of the sphere has been given as a known function $V(\theta)$ (which we will use in the form $V(c)$). First, since nothing singular is happening at the origin, $C_n = 0$ for all n. The A_n are determined by the requirement that ϕ and V agree if $r = R$:

$$V(c) = \phi(R,c) = \sum_{n=0}^{\infty} A_n R^n P_n(c) \quad (5)$$

If we multiply both sides by $P_m(c)$ and integrate c from -1 to 1, we can calculate the lhs (which of course depends on m) and the rhs simplifies because of orthogonality:

$$\int_{-1}^{+1} V(c)P_m(c)\, dc = \sum_{n=0}^{\infty} A_n R^n \int_{-1}^{+1} P_n(c)P_m(c)\, dc = A_m R^m \frac{2}{2m+1} \quad (6)$$

so

$$A_m = \frac{\int_{-1}^{+1} V(c)P_m(c)\, dc}{R^m \frac{2}{2m+1}} \quad (7)$$

For example, if the applied voltage is $+V$ in the northern hemisphere and $-V$ in the southern hemisphere (an odd function of c), we can immediately conclude that for n even $A_n = 0$, and for n odd Mathematica says:

$$A_n R^n \frac{2}{2n+1} = 2V \int_0^{+1} P_n(c)\, dc = \frac{V\sqrt{\pi}}{\Gamma(1-n/2)\,\Gamma((3+n)/2)} \quad (8)$$

```
In[1]:= A=2 Integrate[LegendreP[n,x],{x,0,1}]

                   Sqrt[Pi]
Out[1]= ---------------------------
                   n        3 + n
           Gamma[1 - -] Gamma[-----]
                   2          2
```

Mathematica has provided a complex answer[1] for a result that is just a simple rational number. For your enjoyment, I'll produce a form I can better understand, but in the end we'll let Mathematica use its own result.

I'll begin by reporting some properties of the Gamma function:

$$\Gamma(x+1) = x\Gamma(x) \tag{9}$$

$$\Gamma(n+1) = n! \qquad \text{for } n \text{ a positive integer} \tag{10}$$

$$\Gamma(1-x) = \frac{\pi}{\sin(\pi x)\,\Gamma(x)} \tag{11}$$

$$\Gamma(1/2) = \sqrt{\pi} \tag{12}$$

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{13}$$

The last formula is for the shifted factorial[2] or Pochhammer Symbol defined in class.

Note that n is odd which we will write as $n = 2m-1$, so $m = \{1,2,3,\dots\}$ corresponds to $n = \{1,3,5,\dots\}$.

$$\frac{\sqrt{\pi}}{\Gamma(1-n/2)\,\Gamma((3+n)/2)} = \frac{\sin(\pi n/2)\,\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((3+n)/2)} = \frac{(-1)^{m-1}\,\Gamma(m-1/2)}{\Gamma(1/2)\,\Gamma(m+1)} = \frac{(-1)^m\,2\,\Gamma(m-1/2)}{\Gamma(-1/2))\Gamma(m+1)}$$
$$= (-1)^m \frac{\left(-\frac{1}{2}\right)_m 2}{m!} \tag{14}$$

i.e., $\left\{1, -\frac{1}{4}, \frac{1}{8}, -\frac{5}{64}, \frac{7}{128}, -\frac{21}{512}, \dots\right\}$

Back to Mathematica:

```
f[r_,c_]=Sum[A (2 n +1)/2 r^n LegendreP[n,c],{n,1,21,2}]

ContourPlot[f[Sqrt[x^2+z^2],z/Sqrt[x^2+z^2]],{x,0,.9},{z,-.9,.9},
Contours -> {-.9,-.8,-.7,-.6,-.5,-.4,-.3,-.2,-.1,0,.1,.2,.3,.4,.5,.6,.7,.8,.9},
ContourShading->False,RegionFunction->Function[{x, z, q},x^2+z^2<1 ],
AspectRatio->Automatic]
```

Example 2: Consider the problem of finding ϕ inside and outside a sphere (of radius R) where the surface charge density on the surface of the sphere has been given as a known

[1] Part of the reason for this complex formula is that Mathematica is showing that n even produces zero result. However it doesn't really matter if you don't recognize the answer as Mathematica can quickly produce the rational number for any n you want.

[2] Note: $(1)_n = n!$ more generally $(x)_n$ is n terms multiplied together, starting with x with successive terms one more than the previous.

function $\sigma(\theta)$ (which we will use in the form $\sigma(c)$). First, since nothing singular is happening at the origin, for the inside solution $C_n = 0$ for all n. Since the potential must approach zero as $r \to \infty$, for the outside solution $A_n = 0$ for all n. Thus:

$$\phi(r,\theta) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n r^n P_n(c) & \text{for } r < R \\ \displaystyle\sum_{n=0}^{\infty} \frac{C_n}{r^{n+1}} P_n(c) & \text{for } r > R \end{cases} \tag{15}$$

Continuity of ϕ at $r = R$ produces the requirement:

$$A_n R^n = \frac{C_n}{R^{n+1}} \tag{16}$$

The surface charge density can be related to the discontinuity in the radial component of the electric field:

$$\begin{aligned} \sigma(\theta) &= \epsilon_0 \left(\partial_r \phi\,|_{r=R^-} - \partial_r \phi\,|_{r=R^+}\right) && (17) \\ &= \epsilon_0 \sum_{n=0}^{\infty} \left(n A_n R^{n-1} + (n+1) C_n R^{-(n+2)} \right) P_n(c) && (18) \\ &= \epsilon_0 \sum_{n=0}^{\infty} (2n+1) A_n R^{n-1} P_n(c) && (19) \\ & && (20) \end{aligned}$$

The usual 'Fourier Trick' (multiply both sides by $P_m(c)$ and integrate from -1 to 1 collapsing the sum to a single term) allows A_m to be calculated:

$$\int_{-1}^{+1} \sigma(c) P_m(c)\, dc = \epsilon_0 (2m+1) A_m R^{m-1} \frac{2}{2m+1} = \epsilon_0 2 A_m R^{m-1} \tag{21}$$

Example 3: Often you can calculate ϕ along the z axis, but the off-axis calculation is difficult or impossible. However you can expand $\phi(z)$ to produce the full $\phi(r,c)$ by a trick. Taylor expand $\phi(z)$ to obtain a power series expansion:

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{22}$$

This formula must agree with the Legendre expansion evaluated on the z axis:

$$\phi(r,c) = \sum_{n=0}^{\infty} A_n r^n P_n(c) = \sum_{n=0}^{\infty} a_n z^n \quad \text{on the } z \text{ axis} \tag{23}$$

The fact that on axis $c = \pm 1$ and $P_n(\pm 1) = (\pm 1)^n$ allows easy comparison between these two series. Agreement requires A_n (useful for ϕ off-axis) equals a_n (determined only knowing ϕ on-axis).

For example, the potential on the z-axis for a ring charge (radius R, total charge Q) is clearly

$$\phi(z) = \frac{Q}{4\pi\epsilon_0} \left[z^2 + R^2\right]^{-1/2} = \frac{Q}{4\pi\epsilon_0 R} \left[1 + (z/R)^2\right]^{-1/2} = \frac{Q}{4\pi\epsilon_0 R} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (-z^2/R^2)^n}{n!} \tag{24}$$

we can conclude

$$A_n = \begin{cases} \dfrac{(-1)^{n/2} Q \left(\frac{1}{2}\right)_{n/2}}{4\pi\epsilon_0 \, R^n \, (n/2)!} & \text{for } n \text{ even} \\ \\ 0 & \text{for } n \text{ odd} \end{cases} \tag{25}$$

```
f[r_,c_]=Sum[(-1)^(n/2) Pochhammer[1/2, n/2] r^n LegendreP[n,c]/(n/2)!,{n,0,20,2}]
```

```
ContourPlot[f[Sqrt[x^2+z^2],z/Sqrt[x^2+z^2]],{x,0,.9},{z,-.9,.9},Contours->16,
ContourShading->False,RegionFunction->Function[{x, z, q},x^2+z^2<.8 ],
PlotRangePadding->None,AspectRatio->Automatic]
```

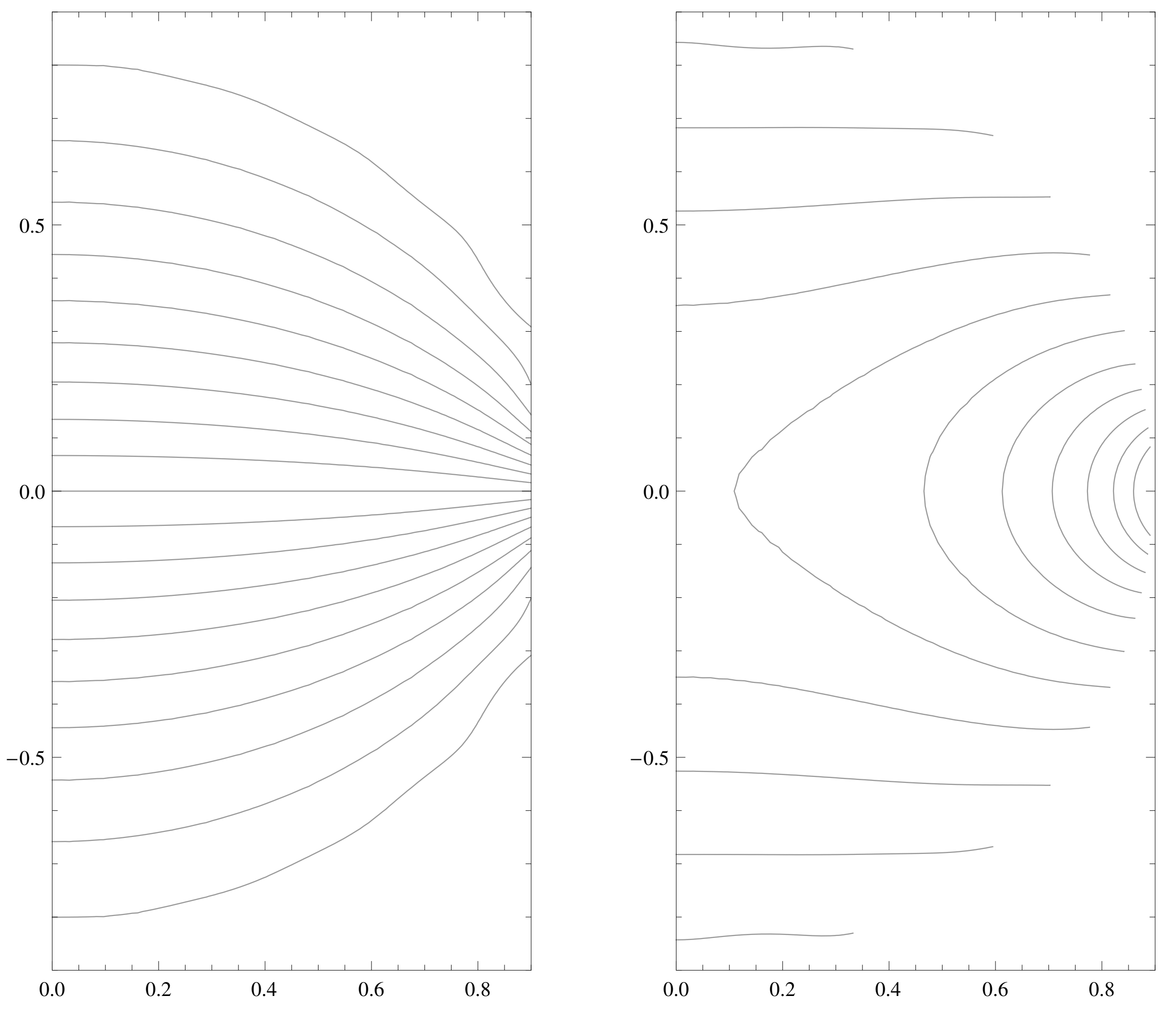


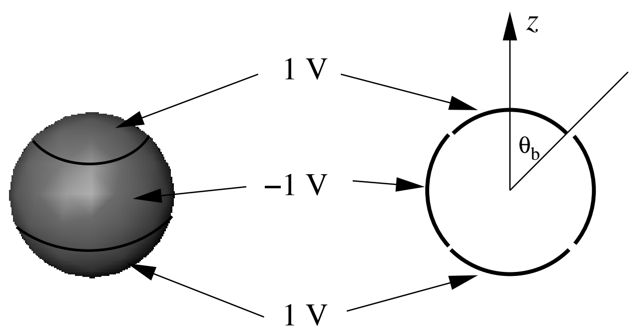
Figure 1: Isopotential contours for Example 1 (left) and Example 3 (right)

Homework 1: A physicist aims to subject a sample to a pure quadrupole field ($n = 2$) inside a spherical cavity. The plan is to charge the top and bottom caps of the sphere to 1 V and the remaining band around the equator to a potential of –1 V. Because the applied voltage $V(\theta)$ is symmetric, terms $A_n = 0$ for n odd. The first important term will then be quadrupole A_2 (A_0 corresponds to a constant voltage and so makes no electric field). It would be nice (but not possible) to make A_2 the only non-zero term. The best we can do is make $A_4 = 0$. Problem: Find the band angle, θ_b that makes $A_4 = 0$. Find A_6 in this circumstance. Find the values: A_0, A_2 and A_6. Put the pieces together to express the potential inside the sphere. Have Mathematica produce a contour plot of that voltage.

Hint:

$$A_4 \propto \int_{-1}^{+1} V(c) P_4(c)\, dc = 2\left\{ -\int_0^{c_b} P_4(c)\, dc + \int_{c_b}^{1} P_4(c)\, dc \right\} \tag{26}$$

Use Mathematica (or a root-finding calculator) to find the value c_b to make this quality zero.



Homework 2: In a region where there are no currents flowing, we can define a magnetic potential that is exactly analogous to the electric potential:

$$\mathbf{B} = -\boldsymbol{\nabla}\phi \qquad \text{where: } \nabla^2\phi = 0 \tag{27}$$

For a pair of Helmholtz coils (two identical coaxial coils with centers separated by R; recall Phys 200 labs with them), the magnetic potential along the axis is given by:

$$\phi(z) = \frac{5\sqrt{5}R}{16}\left[\frac{z - R/2}{\sqrt{R^2 + (z - R/2)^2}} + \frac{z + R/2}{\sqrt{R^2 + (z + R/2)^2}}\right] \tag{28}$$

Recall from the 200 lab that the spacing of the coils is designed to produce a particularly uniform field between the coils, and if you reverse the current in one coil you produce a diverging $\mathbf{B}$ with $\mathbf{B} = 0$ at the center (in short a quadrupole field). Use Mathematica to series expand ϕ around the origin. Use that series to produce $\phi(r, \cos\theta)$ in a region near the origin. Make a contour plot of ϕ to confirm that it is nearly uniform.

If you have reversed coils a distance b above and below the origin, ϕ is given by:

$$\phi(z) = \left[\frac{z - b}{\sqrt{R^2 + (z - b)^2}} - \frac{z + b}{\sqrt{R^2 + (z + b)^2}}\right] \tag{29}$$

What value of b will produce a particularly pure quadrupole field? Make a contour plot of ϕ to confirm that it is nearly quadrupole.

Homework 3: Consider a problem analogous to Helmholtz coils but in electrostatics with charged rings. You have a ring (radius R, centered on the z axis, in a plane parallel to the xy plane) with charge $+Q$ at distance b above the origin, and a similar ring with center at $z = -b$ with charge $-Q$. Find b that will produce the most uniform possible $\mathbf{E}$ field in the vicinity of the origin. Explain why the voltage on the z axis is given by:

$$\phi(z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{R^2+(z-b)^2}} - \frac{1}{\sqrt{R^2+(z+b)^2}}\right] \tag{30}$$

Expand this result in a power series in z. The term linear in z corresponds to $rP_1(c)$ (why?) and further terms produce a non-uniform $\mathbf{E}$. Determine the value of b which makes as many of these further terms zero. Make a contour plot of $\phi(r, \cos\theta)$ (for $R = 1$) to confirm that it is nearly uniform.

Example 4: In the case of cylindrical coordinates where $\phi(r, \theta)$ (and not z), we have:

$$\phi(r,\theta) = A_0 + C_0\ln(r) + \sum_{n=1}^{\infty}\left(A_n r^n + C_n r^{-n}\right)\left(a_n\cos(n\theta) + c_n\sin(n\theta)\right) \tag{31}$$

Note that A_n, C_n, a_n, c_n are not independent: for example you could multiply both A_n, C_n by five and divide both a_n, c_n by five and have exactly the same solution. Most commonly one of the terms in parenthesis is reduced to a single term. As usual we have orthogonality as:

$$\int_{-\pi}^{+\pi}\cos(n\theta)\sin(m\theta)\,d\theta = 0 \tag{32}$$

$$\int_{-\pi}^{+\pi}\cos(n\theta)\cos(m\theta)\,d\theta = \pi\,\delta_{mn} \tag{33}$$

$$\int_{-\pi}^{+\pi}\sin(n\theta)\sin(m\theta)\,d\theta = \pi\,\delta_{mn} \tag{34}$$

We seek ϕ outside a cylinder of radius R on which the potential is known to be

$$V(\theta) = \cos^2(\theta) \tag{35}$$

Since the source extends to infinity, we cannot in general take ϕ at infinity to be zero; thus the meaning of "the potential" on the cylinder is ambiguous; a convenient solution is to define the constant $A_0 = A_0' - C_0\ln(R)$ (a further benefit is the result makes sense dimensionally).

$$\phi(r,\theta) = A_0' + C_0\ln(r/R) + \sum_{n=1}^{\infty}\left(A_n r^n + C_n r^{-n}\right)\left(a_n\cos(n\theta) + c_n\sin(n\theta)\right) \tag{36}$$

Note that in this form the value of C_0 has absolutely no effect on the value of the voltage at $r = R$; A bit of thought should convince you that C_0 is determined by the net charge-per-length on the cylinder. (Recall: voltage for a line charge: $\phi = (-\lambda/2\pi\epsilon_0)\ \ln(r)$.) We will take $C_0 = 0$.

Since the $V(\theta)$ is even in θ, $c_n = 0$; since the electric field should be regular at infinity $A_n = 0$. Thus:

$$\phi(r,\theta) = A_0' + \sum_{n=1}^{\infty} C_n r^{-n}\ \cos(n\theta) \tag{37}$$

Since $\phi(R,\theta)$ must agree with $V(\theta)$ we have:

$$\cos^2(\theta) = \phi(R,\theta) = A_0' + \sum_{n=1}^{\infty} C_n R^{-n} \ \cos(n\theta) \tag{38}$$

The C_n could now be determined using the "Fourier Trick", but a faster way is to use a trig identity to immediately write $\cos^2(\theta)$ in terms of $\cos(n\theta)$:

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\ \cos(2\theta) = A_0' + \sum_{n=1}^{\infty} C_n R^{-n} \ \cos(n\theta) \tag{39}$$

Simple inspection (rather than integration, but of course integration produces the same result):

$$\frac{1}{2} = A_0' \tag{40}$$

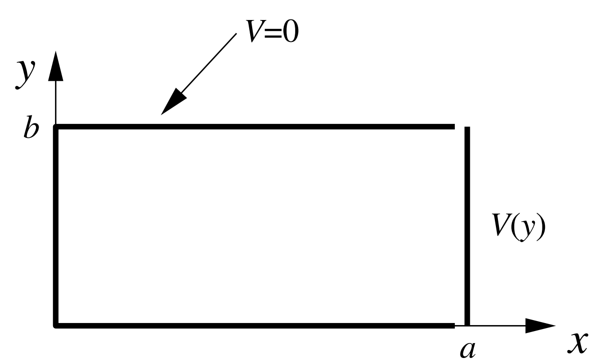
$$0 = C_1 R^{-1} \cos(\theta) \tag{41}$$

$$\frac{1}{2}\ \cos(2\theta) = C_2 R^{-2} \cos(2\theta) \tag{42}$$

and $C_n = 0$ for $n > 2$. So the final result is:

$$\phi(r,\theta) = \frac{1}{2}\left[1 + \cos(2\theta)\ (R/r)^2\right] \tag{43}$$

I hope it is immediately clear that this potential solves Laplace's equation, agrees with $V(\theta)$ when $r = R$ and represents a cylinder with a simple quadrupole.



Example 5: Consider an infinite (in the z directions) rectangular gutter with cross-section between $(0,0)$ and (a,b). Three of the sides of the gutter are grounded; the fourth, (a,y) with $y \in (0,b)$ has a specified voltage $V(y)$. Separation of variables yields a general solution:

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \tag{44}$$

We have orthogonality in the form:

$$\int_0^b \sin\left(\frac{n\pi y}{b}\right)\ \sin\left(\frac{m\pi y}{b}\right)\ dy = \frac{1}{2}\ b\ \delta_{mn} \tag{45}$$

Requiring $\phi(x,y)$ to agree with $V(y)$ when $x = a$ yields:

$$V(y) = \phi(a,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) \tag{46}$$

With the "Fourier Trick" yielding:

$$\int_0^b V(y)\ \sin\left(\frac{m\pi y}{b}\right)\ dy = \frac{1}{2}\, b\, A_m \tag{47}$$

Consider, for example, the case $V(y) = 1$ (constant):

$$\int_0^b \sin\left(\frac{m\pi y}{b}\right)\ dy = \left[\frac{-\cos\left(\frac{m\pi y}{b}\right)}{\frac{m\pi}{b}}\right]_0^b = \frac{b}{m\pi}\ \left[1-(-1)^m\right] \tag{48}$$

So:

$$\phi(x,y) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n}\ \sin\left(\frac{n\pi y}{b}\right) \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \tag{49}$$

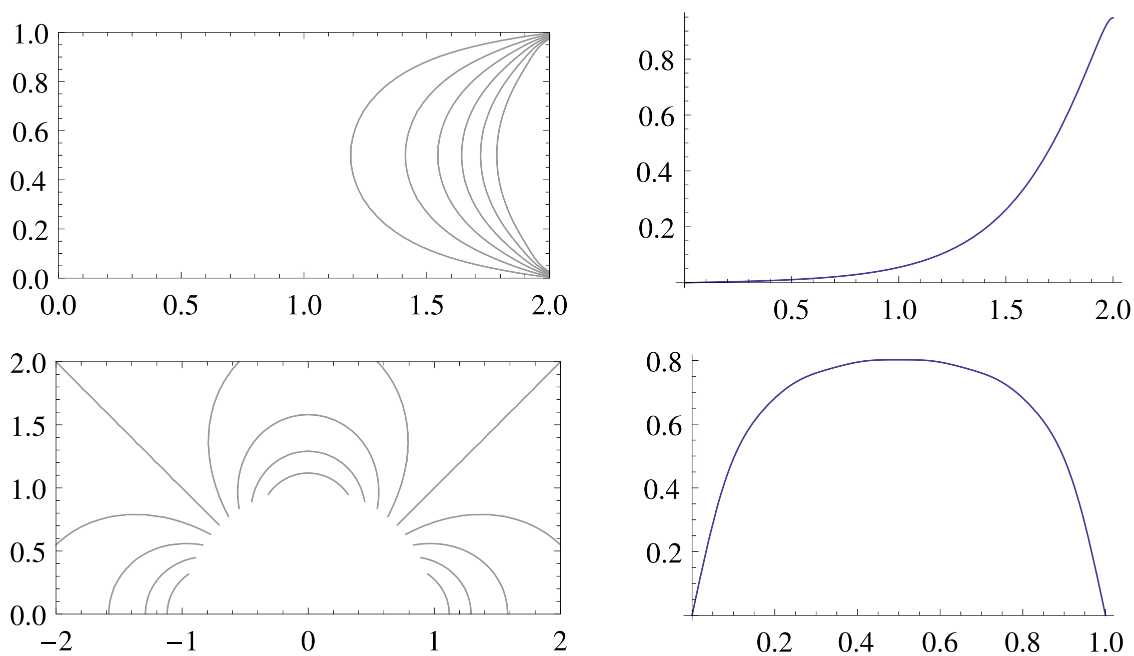


Figure 2: Clockwise from upper left: isopotential contours for Example 5, a slice of Example 5 ϕ along $(x,.5)$, a slice of Example 5 ϕ along $(1.9,y)$, isopotential contours for Example 4

Homework 4: Consider an infinite cylindrical shell of radius $R = 1$, coaxial with the z axis. The voltage on the surface of the shell is given by:

$$V(\theta) = \begin{cases} +1 & \text{for } |\theta| < \pi/4 \\ -1 & \text{for } |\theta - \pi| < \pi/4 \\ 0 & \text{elsewhere} \end{cases} \tag{50}$$

Using 'Fourier's Trick', find the series solution to Laplace's equation inside this cylinder. (Hint: Is this $V(\theta)$ even or odd?)

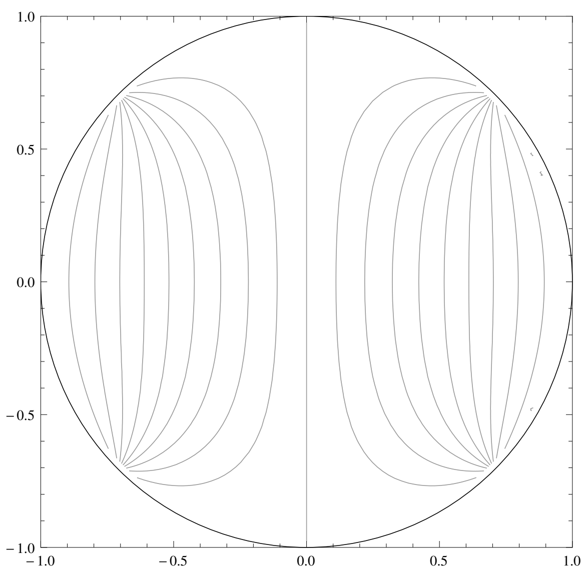
A nice feature of this problem is that Laplace's equation can be solved exactly:

$$\phi(x,y) = \frac{1}{\pi}\left[\arctan\left(\frac{\sqrt{2}\,(x+y)}{1-(x^2+y^2)}\right) + \arctan\left(\frac{\sqrt{2}\,(x-y)}{1-(x^2+y^2)}\right)\right] \tag{51}$$

```
D[D[phi[x,y],x],x]+D[D[phi[x,y],y],y]
Simplify[%]
Out[6]= 0
```

which allows you to compare truncated versions of your infinite series to the exact result. Lets pick the point $\mathbf{p} = (x,y) = (.5,0)$ as a typical point and compare results. Calculate $\phi(\mathbf{p})$ when your sum is truncated to one, two, three, ..., six non-zero terms. Display these results along with the exact result. An easy way to do this is to include the sum-limit in the function definition:

```
phi[x_,y_]=(ArcTan[Sqrt[2](x+y)/(1-(x^2+y^2))]+ArcTan[Sqrt[2](x-y)/(1-(x^2+y^2))])/Pi
phi2[r_,t_,m_]:=Sum[A[[n]]/Pi r^n Cos[n t],{n,1,2 m+1}]
Table[phi2[.5,0,m],{m,0,5}]
Table[phi2[.5,0,m]-phi[.5,0],{m,0,5}]
```



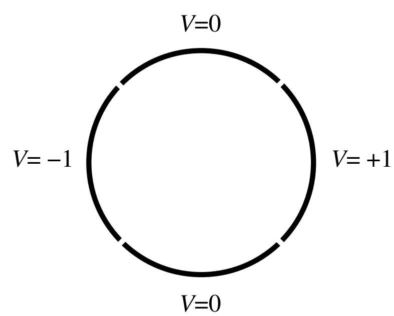


Figure 3: A contour plot of the exact solution (left). A display showing the voltages on the surface of the cylinder (right).