9. Consider the following electric and magnetic fields that come from an oscillating magnetic dipole, m_0 (a wire loop of radius a carrying a current I_0 oscillating at angular frequency ω, $m_0 = \pi a^2 I_0$).

$$\begin{aligned}\vec{\mathbf{E}} &= \frac{m_0 \omega^2 \sin\theta \, e^{i(kr-\omega t)}}{4\pi\epsilon_0 c^3 r}\,\hat{\boldsymbol{\phi}} \\ \vec{\mathbf{B}} &= -\frac{m_0 \mu_0 k^2 \sin\theta \, e^{i(kr-\omega t)}}{4\pi r}\,\hat{\boldsymbol{\theta}}\end{aligned}$$

where $k = \omega/c$ and m_0 are constants. (Note that's kr not $\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}$ and I mean to take the real part of the rhs).

(a) Calculate both sides of the below Maxwell's equations. Are they satisfied?

$$\begin{aligned}\boldsymbol{\nabla}\times\vec{\mathbf{B}} &= \frac{1}{c^2}\frac{\partial\vec{\mathbf{E}}}{\partial t} \\ \boldsymbol{\nabla}\cdot\vec{\mathbf{E}} &= 0\end{aligned}$$

(b) Calculate the time-average Poynting vector ($\langle\vec{\mathbf{S}}\rangle$).

(c) Write down (but don't bother to evaluate) the integral for the total light power leaving a big sphere, and show (by stuffing the integral and other dimensionless terms into an overall factor) that this total power can be expressed:

$$P \propto \sqrt{\frac{\mu_0}{\epsilon_0}}\, I_0^2 \left(\frac{a}{\lambda}\right)^4$$

where λ is the wavelength of the emitted light.

(d) Show that $\sqrt{\frac{\mu_0}{\epsilon_0}}$ has the units of Ω (Ohms). Note that the units of μ_0 and ϵ_0 are given in the previous problem.

(e) Observing at a large distance R from the origin on the positive x axis, the above E and B fields will look a bit like a plane wave:

$$\begin{aligned}\vec{\mathbf{E}} &= \vec{\mathbf{E}}_0 \exp\Big(i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)\Big) \\ \vec{\mathbf{B}} &= \vec{\mathbf{B}}_0 \exp\Big(i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)\Big)\end{aligned}$$

What values of $\vec{\mathbf{E}}_0$, $\vec{\mathbf{B}}_0$, and $\vec{\mathbf{k}}$ (each expressed in rectangular coordinates) produce the best match for the plane wave to the actual wave?

10. A square loop of wire (side s) sits in the xy plane with its center at the origin, sides aligned with the axes. A circular loop of wire (radius R) is to be placed with its center in the xy plane, but with varying orientation (e.g., the plane of the circular loop may be at an angle to the xy plane) and at a distance so large that a well known approximation may be used to calculate the magnetic field due to the square loop. Find the mutual inductance of the two loops if the circular loop is oriented so that the loop lies in the xy plane. How would the loop have to be oriented so that the mutual inductance was zero?