Proofs:

- $\exists\ \vec{A} \ni \vec{B} = \overrightarrow{\nabla \times A}$; $\vec{A}$ = "vector potential"

not unique: gauge transform $\vec{A} \to \vec{A} + \vec{\nabla} f$ produces exactly the same $\vec{B}$

For example:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|}\, dv'$$

↳ this $\vec{A}$ has special property $\vec{\nabla} \cdot \vec{A} = 0$

$\nabla^2 \vec{A} = -\mu_0 \vec{J}$

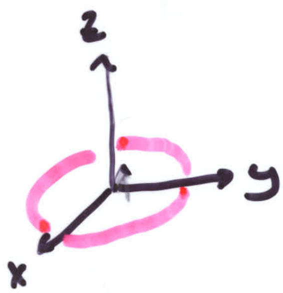
B lines loop

$\overrightarrow{\nabla \times B} = \mu_0 \vec{J} \longrightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}}$

$\vec{\nabla} \cdot \vec{B} = 0 \leftarrow$ no magnetic monopoles

Note: $\overrightarrow{\nabla \times E} = 0 \to$ allows $\vec{E} = -\vec{\nabla}\phi$

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \to$ E lines start on ⊕ end on ⊖



$I\, d\vec{l} = (-\sin\phi, \cos\phi, 0)\, I R\, d\phi$

$\vec{r}' = (R\cos\phi, R\sin\phi, 0)$ ← is A_ϕ

$\vec{r} = (x, 0, z) \to$ only $A_y \neq 0$

$$A_\phi = \frac{\mu_0 I R}{4\pi} \int_{-\pi}^{\pi} \frac{\cos\phi\, d\phi}{\sqrt{\underbrace{x^2 + R^2 + z^2}_{r^2} - 2 \underset{r\sin\theta}{x} R\cos\phi}}$$

$$\xrightarrow[r \gg R]{} \frac{\mu_0 m}{4\pi} \frac{r\sin\theta}{r^3} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$ produces dipole B

m = Area $\cdot\, I$ = "magnetic dipole"

$$\xrightarrow[r \ll R]{} \frac{\mu_0}{4\pi}\, m \frac{r\sin\theta}{R^3}$$ produces $\vec{B} = \frac{\mu_0}{4\pi} \frac{2\vec{m}}{R^3}$

Magnetic Scalar Potential

$$\vec{B} = -\nabla \phi^*$$

μ_0

$$\vec{E} = -\vec{\nabla} \phi$$

voltage

$$\nabla \times B = \mu_0 \vec{J}$$

$$\nabla \times \vec{A} = \vec{B}$$

L

$$\nabla \times \nabla \phi = 0$$

Solid Angles

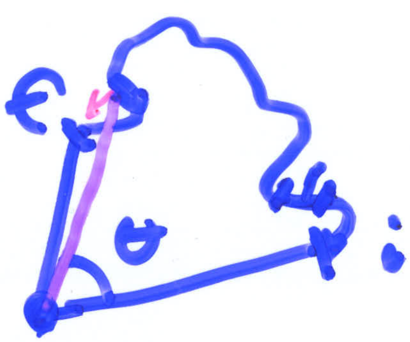
radians

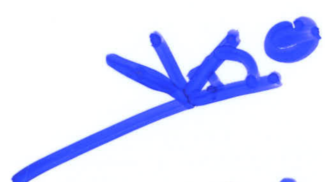
$$\theta = \frac{s}{R}$$

$$\Omega = \frac{A}{R^2}$$

4π sr

all

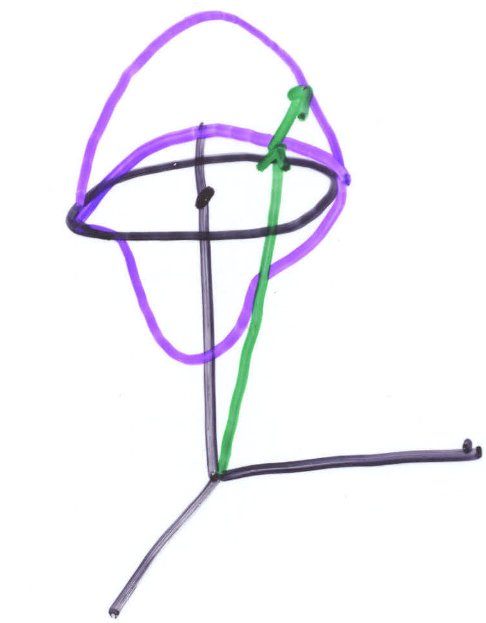




$$\sum \frac{d\ell\, \hat{n} \cdot \vec{r}}{r^2}$$

not zero

$$d\Omega = \frac{\hat{n}\, dA \cdot \vec{r}}{r^3}$$



$$dA = \underbrace{R\,d\theta}_{NS}\;\underbrace{R\sin\theta\,d\phi}_{EW}$$

$$\underset{\uparrow}{\Omega} = \int \frac{dA}{R^2}$$

Omega

$$= \int_0^{\theta} \underbrace{\sin\theta\,d\theta}_{-\cos\theta}\;\underbrace{d\phi}_{2\pi}$$

$$= 2\pi(1-\cos\theta)$$

$\theta = 0$

$\theta = 90^\circ$

$\theta = 180^\circ$

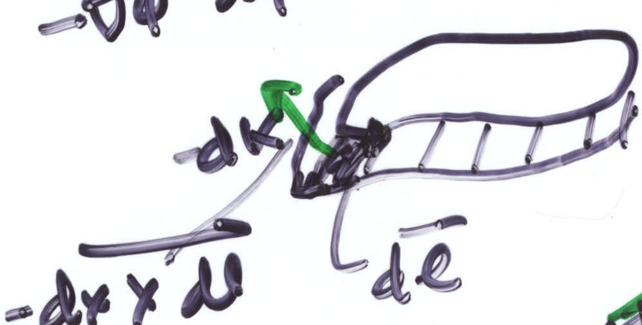
$\phi = \Omega$ to show

$$d\Omega = \vec{B}\cdot d\vec{x}$$

$$\| $$

$$-\nabla\phi\cdot dx$$

$\uparrow d\vec{x}$



$|A\times B| =$

$$d\Omega = \oint \frac{(-d\bar{x}) \times d\bar{l} \cdot \bar{r}'}{r'^3}$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$$

$$= -d\bar{x} \cdot \int \frac{d\bar{l} \times \bar{r}'}{r'^3}$$

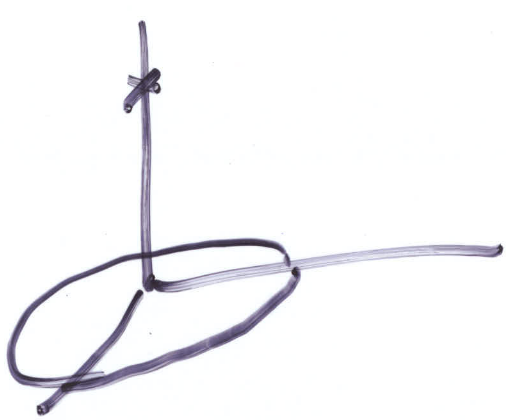
$$\frac{I \mu_0}{4\pi}$$

$$d\Omega = -dx \cdot B \frac{4\pi}{I \mu_0}$$

$$\parallel$$

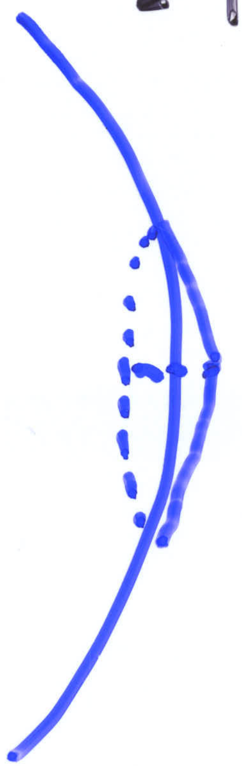
$$\nabla \Omega \cdot dx$$

$$-\nabla \Omega \frac{\mu_0 I}{4\pi} = \bar{B}$$



$$\Omega = 2\pi(1 - \cos\theta)$$

$$\phi = \frac{\mu_0 I}{\cancel{4\pi}\,2} \cancel{2\pi} \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \qquad \frac{z}{\sqrt{z^2 + R^2}}$$



$$\phi = \frac{\mu_0 I}{2}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

$$-\partial_z \phi = \frac{\mu_0 I}{2}\left(\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{2}(z^2+R^2)^{-3/2}\, 2z\, z\right)$$

$$= \frac{\mu_0 I}{2}\left(\frac{1}{\sqrt{}} - \frac{z^2}{\sqrt{}^{\,3}}\right)$$

$$= \frac{\mu_0 I}{2}\left(\frac{z^2+R^2-z^2}{\sqrt{}^{\,3}}\right)$$

$$= \frac{\mu_0 I}{2}\,\frac{R^2}{\sqrt{z^2+R^2}^{\,3}}$$

$$\phi = \frac{p \cdot r}{r^3}$$

$$\boxed{\Omega = \frac{A\,\hat{n}\cdot\vec{r}}{r^3}} = \overbrace{\frac{\vec{m}\cdot\vec{r}}{r^3}}$$

$$-\nabla\Omega = \left(\frac{3(m\cdot\hat{r})\cdot\hat{r} - m}{r^3}\right)$$

$2\pi(1-\cos\theta)$

Multipole expansion → first non zero

dipole $\vec{A} = \frac{m \times r}{r^3}$ ✓

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{J(r') \, dv}{|r - r'|} \quad I \, d\vec{l}$$

$$\uparrow$$

$$d\vec{r}'$$

$$= \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|} \qquad (\alpha)_n$$

$$r >> r' \quad \frac{1}{|r - r'|} = \frac{1}{\sqrt{(r - r') \cdot (r - r')}}$$

$$= \left[r^2 - 2 r \cdot r' + r'^2 \right]^{-1/2}$$

$$= \frac{1}{r} \left[1 - \frac{2 r \cdot r'}{r^2} + \left(\frac{r'}{r}\right)^2 \right]^{-1/2}$$

$$\alpha = \frac{1}{2}$$

$$[1 - x]^{-\alpha} = \sum \frac{(\alpha)_n}{n!} x^n$$

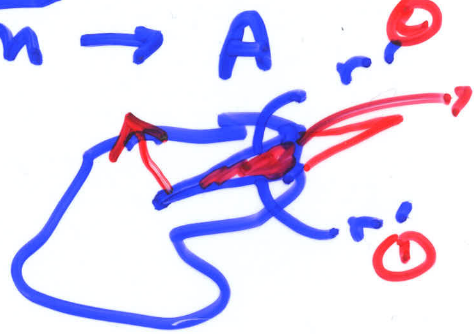
$$= \frac{1}{r} \left[1 + \frac{1}{2} \left(\frac{2 r \cdot r'}{r^2} - \left(\frac{r'}{r}\right)^2 \right) + \cdots \right]$$

small

$$A = \frac{\mu_0 I}{4\pi r} \oint d\vec{r}' \left(\underset{0}{1} + \frac{r \cdot r'}{r^2} \right)$$

$$= \frac{\mu_0 I}{4\pi r^3} \oint d\vec{r}'(r\cdot r')$$

$\vec{m} \to A$



$$\frac{1}{2}\, \underline{r'^{①} \times r'^{②}}$$

$$r'^{②} = r'^{①} + dr'$$

$$r'^{①} \times r'^{①} = 0$$

$$\frac{1}{2}\, r' \times dr'$$

$A \times (B \times C)$

$$\vec{A} = \int \frac{1}{2}\, \vec{r}' \times d\vec{r}'$$

$$dr'(r\cdot r') \to \left(\frac{1}{2}\, \vec{r}' \times d\vec{r}'\right) \times \vec{r}$$

1, 2

$$\| \quad \frac{1}{2}\, \vec{r} \times (dr' \times r')$$

$$\| \quad \frac{1}{2}\left(d\vec{r}'(\vec{r}\cdot\vec{r}') - \vec{r}'(\underline{r\cdot dr'})\right)$$

$$\frac{d(r'(\vec{r}\cdot r'))}{2}$$

$$d\vec{r}'(r\cdot r') = \left(\frac{1}{2}\, \vec{r}' \times dr'\right) \times \vec{r} + \frac{1}{2}\, d(\vec{r}'(r\cdot r'))$$

$\hookrightarrow 0$