1. There is a small spherical (radius R) bubble (containing vacuum) in an otherwise uniform conductor (conductivity g). At large distances from the bubble there is a uniform electric field $\mathbf{E}$ in the z direction. Find the voltage ϕ inside and outside the bubble.

   This is a rather long problem, so let me give you some initial help.

   (a) I claim that the voltage must satisfy Laplace's equation despite all the flowing charge (recall Laplace is replaced by Poisson in the presence of charge density). Prove (explain why) this is the case.

   (b) I claim the following general forms for the voltage inside ($r < R$) and outside ($r > R$) the sphere:

   $$\begin{aligned} \phi_{\text{in}}(r,\theta) &= \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \\ \phi_{\text{out}}(r,\theta) &= \sum_{n=0}^{\infty} C_n r^{-n-1} P_n(\cos\theta) - E_0 z \end{aligned}$$

   where A_n and C_n are currently undetermined constants. Explain why ϕ must have this particular form. What assumptions have been used?

   (c) At the boundary (i.e., $r = R$) the following conditions must hold:

   $$\begin{aligned} \phi_{\text{in}}(R,\theta) &= \phi_{\text{out}}(R,\theta) \\ 0 &= \left.\frac{\partial \phi_{\text{out}}}{\partial r}\right|_{r=R} \end{aligned}$$

   Explain why these particular conditions must hold.

   (d) I conclude from the first condition that, for $n \neq 1$:

   $$A_n R^{2n+1} = C_n$$

   Explain the basis for this conclusion and report the analogous result for $n = 1$.

   (e) From the second boundary condition I conclude that the coefficient of $P_n(\cos\theta)$ in ϕ_{out} must be zero for every n. What is the coefficient of $P_1(\cos\theta)$? (Careful)

   (f) Find the electric field inside the bubble.

   (g) I claim the radial component of the electric field is not continuous across the bubble boundary, i.e,:

   $$\left.\frac{\partial \phi_{\text{in}}}{\partial r}\right|_{r=R} \neq \left.\frac{\partial \phi_{\text{out}}}{\partial r}\right|_{r=R}$$

   What is the physical (in contrast to mathematical) cause of this discontinuity?

2. The $z = 0$ plane is the interface between two materials: the region of space with $z > 0$ is vacuum; the region with $z < 0$ is a dielectric with $\epsilon = 5\epsilon_0$. The interface carries a free surface charge density $\sigma_f = 10^{-6}\,\mathrm{C/m^2}$. On the vacuum side of the interface ($z = 0^+$): $\vec{\mathbf{E}} = \langle 10^2, 0, 10^3 \rangle$ V/m. Find $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ in the dielectric at $z = 0^-$. ($\epsilon_0 = 8.85 \times 10^{-12} \quad \mathrm{C^2/(N \cdot m^2)}$)

3. The space between two concentric, spherical, conducting shells of radii a and b is filled with a dielectric with dielectric constant K. The inner conductor (a) carries a total free charge Q; The outer conductor (b) carries a total free charge $-Q$. Find $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ between the shells. What is the electric field inside a and outside b?. Find the total electrostatic energy by appropriately integrating $\vec{\mathbf{E}} \cdot \vec{\mathbf{D}}$ over all space. We can select the voltage on the outer conductor (b) to be zero. What is then the voltage on the inner conductor (a)? Find the total electrostatic energy by appropriately multiplying voltages times charges.

4. A steady current I flows down an infinite cylindrical wire of radius R in such a way that the current density is uniform throughout the wire's cross-section. Find $\vec{\mathbf{B}}$ everywhere in space (i.e., both $r < R$ and $r > R$). Hint: Ampere's law.

5. A circular loop of wire (radius R) sits in the xy plane with its center at the origin. The loop carries a current I flowing in the counter-clockwise direction as seen from above (i.e., $z > 0$). Consider an attempt to calculate the resulting magnetic field in the xy plane a distance d from the coil center using the formula:

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi} \int I d\vec{\boldsymbol{\ell}}' \times \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}$$

Report expressions for all of the following: $d\vec{\boldsymbol{\ell}}'$, $\vec{\mathbf{r}}$, $\vec{\mathbf{r}}'$, $|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|$. Calculate the cross product. Write down (but do not try to calculate) the resulting integral. Carefully explain how you have used the symmetry of the problem.