Class 7: old exam 2: 2, 3, 4, 7.14, 7.29, 7.40

2) $x_1 = L\sin\theta_1$ $\qquad \dot{x}_1 = L\dot{\theta}_1\cos\theta_1$
$y_1 = -L\cos\theta_1$ $\qquad \dot{y}_1 = L\dot{\theta}_1\sin\theta_1$ $\Big) \quad \dot{x}_1^2+\dot{y}_1^2 = (L\dot{\theta}_1)^2$

$x_2 = L(\sin\theta_1 + \sin\theta_2)$ $\qquad \dot{x}_2 = L(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)$

$y_2 = -L(\cos\theta_1 + \cos\theta_2)$ $\qquad \dot{y}_2 = L(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)$

$$\dot{x}_2^2+\dot{y}_2^2 = L^2\left\{\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\underbrace{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}_{\cos(\theta_2-\theta_1)}\right\}$$

$$KE = \frac{1}{2}mL^2\left[\dot{\theta}_1^2+\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right]$$

$$PE = -mgL\left[\cos\theta_1+\cos\theta_1+\cos\theta_2\right] = -mgL\left[2\cos\theta_1+\cos\theta_2\right]$$

$$\frac{L}{mL^2} = \frac{1}{2}\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right]+\frac{g}{L}\left[2\cos\theta_1+\cos\theta_2\right]$$

$$\frac{\partial L}{\partial\theta_1} = \dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - \frac{2g}{L}\sin(\theta_1) = \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_1} = 2\ddot{\theta}_1 + \cancel{\ } + (\dot{\theta}_2\cos(\theta_2-\theta_1))^{\bullet}$$

$$\frac{\partial L}{\partial\theta_2} = -\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - \frac{g}{L}\sin(\theta_2) = \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_2} = \ddot{\theta}_2 + (\dot{\theta}_1\cos(\theta_2-\theta_1))^{\bullet}$$

3) $f(x) = g\cos(x/R)$ $\qquad f(x)=0 \rightarrow \frac{x}{R} = \pm\pi/2,\ \pm\frac{3\pi}{2}$, etc

$\pi/2$: + stable, − unstable; $3\pi/2$: + unstable, − stable

$$\omega^2 = -f'(x_0) = \frac{g}{R}\sin\left(\frac{x}{R}\right)\Big|_{\pi/2} = \frac{g}{R}$$

4) a) $PE = \frac{1}{2}kx^2 - mgx$

b) $KE = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2$

c) $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I}{R^2}\dot{x}^2 + mgx - \frac{1}{2}kx^2$

$$\frac{\partial L}{\partial x} = -kx+mg = m(1+I/mR^2)\ddot{x}$$

d) $-kx+mg = 0$ if $x = \frac{mg}{k}$ & $f' = -k$

e: $$\omega^2 = \frac{k}{m(1+I/mR^2)}$$

7.14, 7.29, 7.40

7.14) cylinder $I = \frac{1}{2}MR^2$; amount of unrolling ϕ related to altitude $y = R\phi$

$$KE = KE_{cm} + KE_{spin} = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\underbrace{I}_{\frac{1}{2}MR^2}\omega^2 \leftarrow \dot{y}/R$$

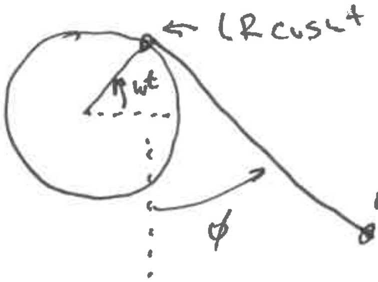
$$= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}m\dot{y}^2 = \frac{3}{4}m\dot{y}^2$$

$PE = -mgy$

$$L = \frac{3}{4}m\dot{y}^2 + mgy \rightarrow \frac{\partial L}{\partial y} = mg = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\frac{3}{2}m\dot{y} = \frac{3}{2}m\ddot{y}$$

$$\frac{2}{3}g = \ddot{y}$$

7.29



$\leftarrow (R\cos\omega t, R\sin\omega t)$

relative: $(l\sin\phi, -l\cos\phi)$

total: $(R\cos\omega t + l\sin\phi,\ R\sin\omega t - l\cos\phi)$

$$\dot{\vec{r}} = (-R\omega\sin\omega t + l\dot{\phi}\cos\phi,\ R\omega\cos\omega t + l\dot{\phi}\sin\phi)$$

$$v^2 = \dot{\vec{r}}^2 = (-R\omega\sin\omega t + l\dot{\phi}\cos\phi)^2 + (R\omega\cos\omega t + l\dot{\phi}\sin\phi)^2$$

$$= R^2\omega^2(\sin^2\omega t + \cos^2\omega t) + (l\dot{\phi})^2(\cos^2\phi + \sin^2\phi)$$
$$+ 2R\omega l\dot{\phi}\,[\cos\omega t\sin\phi - \sin\omega t\cos\phi]$$
$$\hookrightarrow -\sin(\omega t - \phi)$$

$$= R^2\omega^2 + (l\dot{\phi})^2 - 2R\omega l\dot{\phi}\sin(\omega t - \phi)$$

$$U = mg(R\sin\omega t - l\cos\phi)$$

$$L = \frac{1}{2}m\left[R^2\omega^2 + (l\dot{\phi})^2 - 2R\omega l\dot{\phi}\sin(\omega t - \phi)\right] + mg(l\cos\phi - R\sin\omega t)$$

$$\frac{\partial L}{\partial \phi} = mR\omega l\dot{\phi}\cos(\omega t - \phi) - mgl\sin\phi = \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{d}{dt}\left[ml^2\dot{\phi} - mR\omega l\cancel{\dot{\phi}}\sin(\omega t - \phi)\right]$$

$$= ml^2\ddot{\phi} - mRl\omega\cos(\omega t - \phi)(\omega - \dot{\phi})\ \cancel{\quad}$$

cancel

$$-mgl\sin\phi = ml^2\ddot{\phi} - mRl\omega^2\cos(\omega t - \phi)$$

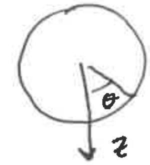
$$\underbrace{\frac{R}{l}\omega^2\cos(\omega t - \phi)}_{\text{like a driven oscillator}} = \ddot{\phi} + \frac{g}{l}\sin\phi$$

but with $\omega t - \phi$ not just ωt

7.20;

$\vec{v} = R\dot{\theta}\,\hat{\theta} + R\sin\theta\,\dot{\varphi}\,\hat{\varphi}$

$v^2 = (R\dot{\theta})^2 + (R\sin\theta\,\dot{\varphi})^2$



$$L = \frac{1}{2} m R^2 \left[\dot{\theta}^2 + (\sin\theta\,\dot{\varphi})^2 \right] + mgR\cos\theta$$

divide by mR^2 define $\omega^2 = \frac{g}{R}$ $\rightarrow$ $L = \frac{1}{2}\left[\dot{\theta}^2 + (\sin\theta\,\dot{\varphi})^2\right] + \omega^2\cos\theta$

No φ so $P_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \sin^2\theta\,\dot{\varphi} = $ Constan

Note: $\frac{\vec{l}}{m} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \\ R & 0 & 0 \\ 0 & R\dot{\theta} & R\sin\theta\dot{\varphi} \end{vmatrix} = -R^2\sin\theta\,\dot{\varphi}\,\hat{\theta} + R^2\dot{\theta}\,\hat{\varphi}$

$\hat{\theta}\cdot\hat{k} = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)\cdot(0,0,1) = -\sin\theta$

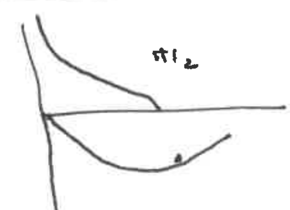
so $\frac{l_z}{m} = R^2\sin^2\theta\,\dot{\varphi} \rightarrow$ the conserved P_φ above

$$\frac{\partial L}{\partial\theta} = -\omega^2\sin\theta + \sin\theta\cos\theta\,\dot{\varphi}^2 = \ddot{\theta}$$

$\dot{\varphi} = \frac{P_\varphi}{\sin^2\theta}$

$$= \underbrace{-\omega^2\sin\theta + \frac{\cos\theta}{\sin^3\theta}P_\varphi^2}_{} = \ddot{\theta}$$

must equal zero in range $0 \to \pi/2$



negative @ $\pi/2$ } between a root
∞ @ $\theta = 0$

$f(\theta) = 0 \leftrightarrow \sin\theta = \left(\frac{P_\varphi}{\omega}\right)^2 \frac{\cos\theta}{\sin^3\theta}$ $\rightarrow$ result: circle motion at Const θ

↳ not easy - if $c = \cos\theta$ requires

$\sin^4\theta = (1-c^2)^2 = c\left(\frac{P_\varphi}{\omega}\right)^2$

osc freq: $-f'(\theta_0)$

Remark: if $P_\varphi \gg \omega$ then $c \sim \left(\frac{\omega}{P_\varphi}\right)$

in order to be <1 ; ie $\theta \sim \pi/2$

In reverse limit $\theta \sim \sqrt{\frac{P_\varphi}{\omega}}$