Complete 4 problems

1. Consider the Lagrangian (m and B are constants)

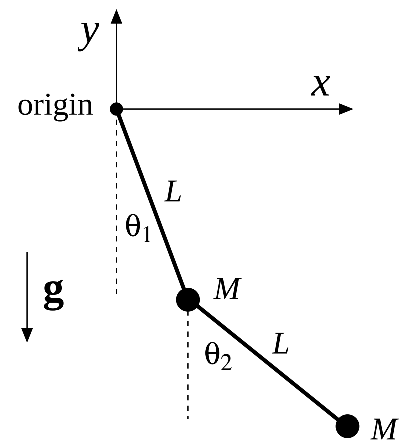
$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + Bx\dot{x} - V(x,y)$$

   (a) Using the Lagrangian, find the differential equations (Euler-Lagrange equations) describing the motion of the particle.

   (b) Find the generalized (a.k.a., canonical) momenta p_x and p_y

   (c) Find the Hamiltonian.

2. Consider the equal-mass (M), equal-length (L), double pendulum. The pendulum consists of two identical pendulums with the second pendulum attached to the first.



   (a) Write down the (x,y) locations of the two masses: $\mathbf{r}_1$ & $\mathbf{r}_2$ in terms of the coordinates θ_1 & θ_2.

   (b) Calculate the velocities: $\mathbf{v}_1$ and $\mathbf{v}_2$. Don't bother to simplify.

   (c) With some trig identities the total kinetic energy can be written as:

$$\text{KE} = \frac{1}{2}\,ML^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right)$$

   Write down the total potential energy and the resulting Lagrangian.

   (d) Find the equations of motion for θ_1 and θ_2. Don't bother to simplify.

3. Consider the equation of motion:

$$\ddot{x} = g\cos(x/R)$$

   where g, R are positive constants.

   (a) Report an unstable equilibrium point (i.e., a solution: $x(t)$ =constant which, while possible, is unstable).

   (b) Report a stable equilibrium point (i.e., a solution: $x(t)$ =constant which is stable).

   (c) Show the steps required to find the frequency of small oscillations about the stable equilibrium point. Report the resulting frequency of small oscillations.

4. As shown left, the extension (x) of a spring beyond its relaxed length, controls the altitude of a mass, m, through a massless string which is looped over a frictionless pulley (with radius R and moment of inertia I). The acceleration of gravity is represented by the positive constant g; the spring has spring constant k.

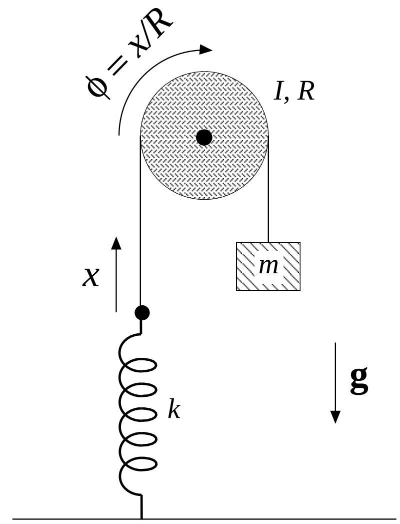
   (a) Write down the total potential energy as a function of x.

   (b) Write down the total kinetic energy as a function of $\dot{x}$.

   (c) Using the Lagrangian, write down the equation of motion of this system.

   (d) The system has a stable equilibrium point; Report it!

   (e) Find the frequency of small oscillations about the stable equilibrium point.



5. Consider the motion of two particles (with masses m_1 and m_2) where the only forces present are due to an interaction potential that just depends of the distance between the two particles: $U(|\mathbf{r}_1 - \mathbf{r}_2|)$. The Lagrangian for this system would be:

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|)$$

However we reduced this problem to just:

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\left(\dot{r}^2 + (r\dot{\phi})^2\right) - U(r)$$

where $\mathbf{R}$ is the location of the center of mass and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ was now described by the 2d polar coordinates $r = |\mathbf{r}|$ and ϕ. And then just one differential equation told much of the story:

$$\mu\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{\ell^2}{2\mu r^2} + U(r)\right)$$

where ℓ is the (constant) angular momentum. Describe in detail how this reduction from six coordinates to just one important differential equation was accomplished (e.g., what conditions were required to make this reduction possible). Some things you will want to include:

   (a) What is the equation of motion of the center of mass? In what sense is the result 'trivial' and 'expected'?

   (b) What is the formula for the total (i.e., orbital and spin) angular momentum? How can you be sure that it is conserved? Are the orbital angular momentum and spin angular momentum separately conserved? Why or why not?

   (c) $\mathbf{r}$ has been expressed in terms of 2d polar coordinates rather than 3d spherical coordinates because the motion must be in a plane. What evidence assures that $\mathbf{r}$ remains in a plane?

   (d) What is the equation of motion for ϕ? How has it been included in the differential equation?