Thomas Precession

1 Discussion

In 1926 physicists were busy using the newly developed quantum mechanics to explain the behavior of electrons "orbiting" around the nucleus. Many pieces fell quickly into place, but there remained puzzling oddities which we now know are answered by proper application of special relativity. Relativity was a somewhat surprising source of solution since the speed of the electron in, say, a hydrogen atom is "only" about $\beta = v/c = \frac{1}{137}$. Most relativistic effects are second-order in β (e.g., $\gamma = 1/\sqrt{1-\beta^2} \approx 1 + \frac{1}{2}\beta^2 + \cdots$) and so "small" effects were expected whereas the problems had to do with factor-of-two shifts. L.H. Thomas (1903–1992) is correctly famous for his solution to one of these problems which involves a frame-rotation effect in centripetally accelerated, fast-moving particles. We will make no attempt to put "Thomas precession" into its atomic context, and instead aim to show that a series of Lorentz transformations that seemingly should produce no effect, in fact produces a frame rotation.

I assume you are familiar with the following aspects of special relativity:

1. If we make a "boost", e.g., jump to a frame of reference moving along the x-axis at a speed given by β, then coordinates in the new frame (x', y', z', ct') can be related to the coordinates in the original frame (x, y, z, ct) by a matrix equation:

$$\begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix}$$

2. 4-tuples like: (x, y, z, ict) that transform according to the above matrix equation are called four-vectors. Yes, i in the 4^{th} component is $\sqrt{-1}$. Use of this "complex metric" will make our life easier (even though the "complex metric" is, in turn, replaced by seemingly more difficult, but real-valued, covariant/contravariant entities in more advanced treatments of relativity). Another important four-vector is the velocity four-vector:

$$u = \gamma(\beta_x, \beta_y, \beta_z; i) = \gamma(\vec{\boldsymbol{\beta}}; i)$$

   Note that:

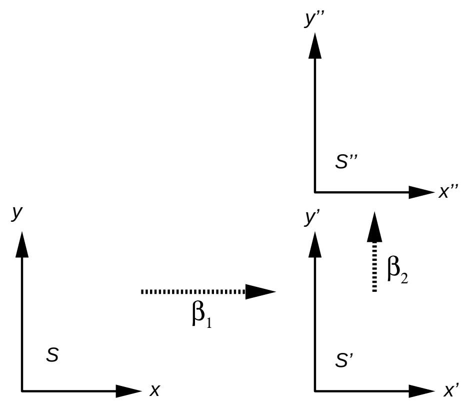
$$u^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 = \gamma^2(\vec{\boldsymbol{\beta}}^2 - 1) = -1$$

   Additionally, since γ is a common factor for all four components, we can determine $\beta_x = iu_1/u_4$ or more generally: $\vec{\boldsymbol{\beta}} = i\vec{u}/u_4$.

3. Simple rotation of the frames is also described by matrix transformations. For example, if the primed frame is rotated by an angle θ in 2D we have:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

   Note: both the Lorentz transformation and the simple rotation leave invariant the square of vectors. This property defines the matrices as orthogonal matrices.



Frame S is our original frame. Frame S' is moving at a velocity given by $\vec{\beta}_1$ relative to S and frame S'' is moving at a velocity given by $\vec{\beta}_2$ relative to S'. I've displayed the matrix for a boost along the x-axis; *Mathematica* can supply us with the (quite complicated) matrix for boosts in an arbitrary direction: `boost[bx,by,bz]` if we include a package:

```
In[1]:= <<LorentzBoost.m
```
. . . Learn the function `boost[bx,by,bz]`

```
In[2]:= m=boost[0,b2,0].boost[b1,0,0]
```
. . . `m` is the matrix which first boosts along x with β=`b1` to S' and then boosts along y' with β=`b2` to S''.

To figure out how to boost back to S, consider the four-velocity of the origin of S. The origin of S is not moving as viewed from S, so in S its 4-velocity is $u = (0, 0, 0, i)$. If we transform u to see what u is in S'' we find:

```
In[3]:= u2=m.{0,0,0,I}
```

We now have the velocity of the origin of S as seen in S''. If we were to boost to exactly that velocity, the velocity of the origin of S in this new frame would be zero. Thus we determine that velocity and boost back to a rest frame for the origin of S. It turns out that this rest frame for the origin of S, arrived at via a triangle of pure boosts, is rotated from S.

```
In[4]:= boost[I u2[[1]]/u2[[4]],I u2[[2]]/u2[[4]],I u2[[3]]/u2[[4]]].m
In[5]:= n=Simplify[%]
```
. . . `Simplify` helped a lot but it's still a mess

It's not as bad as it looks; part of the problem is that *Mathematica* has problems simplifying roots. Notice the last matrix entry: `n[[4,4]]`:

```
            2                2
Sqrt[1 - b1 ] Sqrt[1 - b2 ]
---------------------------
              2            2
Sqrt[(-1 + b1 ) (-1 + b2 )]
```

This is 1, but *Mathematica* doesn't see it. With some diligent work you could show that the structure of this matrix is:

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

That is:

`n[[1,1]]=n[[2,2]]` $= a$ and `n[[1,2]]=-n[[2,1]]` $= b$

This means that the net result of these 3 boosts is a simple rotation in the xy plane by an angle $\theta = \sin^{-1}(b)$. We can see the structure of the matrix better if we look just at the terms relevant for "small" $\vec{\boldsymbol{\beta}}_i$. In the atomic application $\beta_1 \approx \frac{1}{137}$, and $\vec{\boldsymbol{\beta}}_2$ is the small change in velocity due to the electric attraction to the nucleus during some time dt. Thus is makes sense to expand for small β_i:

`In[6]:= Series[%,{b1,0,1},{b2,0,1}]`	...Does a Taylor expansion
`In[7]:= nsmall=Normal[%]`	...turns the series back into a normal polynomial

The result is:

$$\begin{bmatrix} 1 & -\frac{1}{2}\beta_1\beta_2 & 0 & 0 \\ \frac{1}{2}\beta_1\beta_2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comparison with our rotation matrix shows the result[1] is a small rotation in xy plane: $\theta \approx -\frac{1}{2}\beta_1\beta_2$ (we assume: $\sin\theta \approx \theta, \cos\theta \approx 1$). The exact value of θ can be calculated from `ArcSin[n[[1,2]]]`.

2 Homework

Follow the above method, but use velocities first in the y direction and then in the z direction rather than x, y. As above find the rotation axis and magnitude (for small β).

Find the exact rotation angle in the following five cases: $\beta_y = \beta_z = .99$, $\beta_y = \beta_z = .9$, $\beta_y = \beta_z = .1$, $\beta_y = .99$ & $\beta_z = .1$, and $\beta_y = .1$ & $\beta_z = .99$.

Using a `boost` with an arbitrary $\vec{\boldsymbol{\beta}}$, show that the determinant of the boost matrix is 1, and that the boost matrix is orthogonal:

$$O^{\mathrm{T}} \cdot O = 1$$

i.e., that the transpose of the matrix is the inverse of the matrix. In short: the matrix is in $SO(4)$.

Turn in a printout showing each step as *Mathematica* solves the problem.

[1] In the atomic situation β_2 is the result of acceleration during an infinitesimal interval dt. So we can calculate a rate of rotation $\omega = \beta_1 a/2c$