King Oscar's Prize

1 Discussion

> Given a system of arbitrarily many mass points that attract each other according to Newton's laws, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time and for all whose values the series converges uniformly.
>
> This problem, whose solution would considerably extend our understanding of the solar system, would seem capable of solution using analytic methods presently at our disposal; we can at least suppose as much since Lejeune Dirichlet communicated shortly before his death to [Leopold Kronecker], that he had discovered a method for integrating the differential equations of Mechanics, and that by applying this method, he had succeeded in demonstrating the stability of our planetary system in an absolutely rigorous manner. Unfortunately, we know nothing about this method [because Dirichlet died before writing it down], except the theory of small oscillations would appear to have served as his point of departure for this discovery. We can nevertheless suppose, almost with certainty, that this method was based not on long and complicated calculations, but on the development of a fundamental and simple idea that one could reasonably recover through preserving and penetrating research.
>
> *Acta Mathematica*, vol. 7, 1885–86

It is a familiar story: find the lost proof of a dead mathematician. However in this case there was a new twist: King Oscar II of Sweden and Norway had been convinced by Gösta Mittag-Leffler to award a modest cash prize and much honor on the person who could prove the stability of the solar system in time for his 60^{th} birthday.

Folklore connects Mittag-Leffler and King Oscar's prize to Alfred Nobel's prizes (first awarded in 1901). It is often said (incorrectly) that the reason there is no Nobel Prize in mathematics is that Mittag-Leffler had an affair with Nobel's wife. This is clearly false (since Nobel did not marry) and in fact while Nobel was born in Stockholm, he lived much of his adult life in Paris. While Nobel left no record explaining his choice in prizes (in physics, chemistry, physiology or medicine, literature and peace), it is clear he neglected mathematics because he did not think of mathematics as a major source of inventions that benefited mankind. It is interesting to try to guess how Mittag-Leffler convinced King Oscar that this problem *was* deserving of the King's notice. Whatever was promised, it is clear that the prize-winning essay did not deliver it, and we now know that Mittag-Leffler was soon engaged in a quite successful cover-up of the botched prize-winning essay. Oddly enough, about 25 years after this competition the Finish astronomer Karl F. Sundman (1873–1947) solved the King's problem (for the case of three interacting bodies), but his solution—a convergent series—is of <u>no</u> practical value. (That is, it in no way "considerably extend[s] our understanding of the solar system".) On the other hand the revised prize-winning essay (which came the the opposite conclusion as the botched essay, which "won" the prize), presents nothing like the requested convergent series, and instead is the first herald of chaos in Newtonian mechanics.

The subject of King Oscar's prize: the N-body problem, involves solving the differential equations: $\mathbf{F}_i = m_i \mathbf{a}_i$ for each of the N bodies, where the force on the i^{th} body, $\mathbf{F}_i$, is the total gravitational force due to the other $N-1$ bodies, m_i is the mass of the i^{th} body, and $\mathbf{a}_i$ is the acceleration of the i^{th} body: $\mathbf{a}_i = d^2\mathbf{r}_i/dt^2$. The attractive force of the j^{th} body on the i^{th} body, points in the direction of the j^{th} body, and, in magnitude, falls off with the inverse square of the distance between the two bodies, $|\mathbf{r}_j - \mathbf{r}_i|$:

$$\mathbf{F}_{ij} = \frac{G m_j m_i (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

(At first glance this may appear to be inverse cube in distance, but notice that the magnitude of the vector in the numerator cancels a factor of distance in the denominator.) The total force on the i^{th}

body, $\mathbf{F}_i$, is the vector sum of the forces acting on that body:

$$m_i\mathbf{a}_i = \mathbf{F}_i = \sum_{j=1}^{N}{}' \, \mathbf{F}_{ij} = \sum_{j=1}^{N}{}' \, \frac{Gm_j m_i(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

or

$$\frac{d^2\mathbf{r}_i}{dt^2} = \sum_{j=1}^{N}{}' \, \frac{Gm_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

where Σ' means the sum excludes the term where $j = i$.

Thus for the first body of three interacting bodies (N=3) we have:

$$\frac{d^2\mathbf{r}_1}{dt^2} = \frac{Gm_2(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{Gm_3(\mathbf{r}_3 - \mathbf{r}_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3}$$

Note that this is a vector equation, so it is equivalent to three equations, one for each component:

$$\begin{aligned}
\frac{d^2x_1}{dt^2} &= \frac{Gm_2(x_2 - x_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{Gm_3(x_3 - x_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \\
\frac{d^2y_1}{dt^2} &= \frac{Gm_2(y_2 - y_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{Gm_3(y_3 - y_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} \\
\frac{d^2z_1}{dt^2} &= \frac{Gm_2(z_2 - z_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + \frac{Gm_3(z_3 - z_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3}
\end{aligned}$$

In the *restricted* 3-body problem, we consider motions that remain forever in the $z = 0$ plane, so only x and y can vary. Thus for the first body we have 2 coupled differential equations:

$$\begin{aligned}
\frac{d^2x_1}{dt^2} &= \frac{Gm_2(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + \frac{Gm_3(x_3 - x_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3} \\
\frac{d^2y_1}{dt^2} &= \frac{Gm_2(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}^3} + \frac{Gm_3(y_3 - y_1)}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}^3}
\end{aligned}$$

There are, of course, a similar pair of differential equation for the second and third bodies.

It is always easier to solve differential equations if we eliminate the units. Thus we will write masses in terms of a to-be-determined standard mass M: $m_i = m_i' M$ where m_i' is itself dimensionless, time in terms of a to-be-determined standard time T: $t = t'T$ and all positions in terms of a standard length L, for example: $x_i = x_i' L$. Thus:

$$\begin{aligned}
\frac{L}{T^2}\frac{d^2x_1'}{dt'^2} &= \frac{M}{L^2}\left(\frac{Gm_2'(x_2' - x_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}^3} + \frac{Gm_3'(x_3' - x_1')}{\sqrt{(x_3' - x_1')^2 + (y_3' - y_1')^2}^3}\right) \\
\frac{L}{T^2}\frac{d^2y_1'}{dt'^2} &= \frac{M}{L^2}\left(\frac{Gm_2'(y_2' - y_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}^3} + \frac{Gm_3'(y_3' - y_1')}{\sqrt{(x_3' - x_1')^2 + (y_3' - y_1')^2}^3}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{d^2x_1'}{dt'^2} &= \frac{GT^2M}{L^3}\left(\frac{m_2'(x_2' - x_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}^3} + \frac{m_3'(x_3' - x_1')}{\sqrt{(x_3' - x_1')^2 + (y_3' - y_1')^2}^3}\right) \\
\frac{d^2y_1'}{dt'^2} &= \frac{GT^2M}{L^3}\left(\frac{m_2'(y_2' - y_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}^3} + \frac{m_3'(y_3' - y_1')}{\sqrt{(x_3' - x_1')^2 + (y_3' - y_1')^2}^3}\right)
\end{aligned}$$

Now all the terms in the equation are unitless including GT^2M/L^3. If we choose our standard units cleverly, we can make this last term one. Thus if T=1 year, M=Sun's mass, and L=1 AU (AU =

astronomical unit = distance between Earth and Sun), GT^2M/L^3=1. Our final equations for the first body are then:

$$
\begin{aligned}
\frac{d^2x'_1}{dt'^2} &= \frac{m'_2(x'_2-x'_1)}{\sqrt{(x'_2-x'_1)^2+(y'_2-y'_1)^2}^3} + \frac{m'_3(x'_3-x'_1)}{\sqrt{(x'_3-x'_1)^2+(y'_3-y'_1)^2}^3} \\
\frac{d^2y'_1}{dt'^2} &= \frac{m'_2(y'_2-y'_1)}{\sqrt{(x'_2-x'_1)^2+(y'_2-y'_1)^2}^3} + \frac{m'_3(y'_3-y'_1)}{\sqrt{(x'_3-x'_1)^2+(y'_3-y'_1)^2}^3}
\end{aligned}
$$

You should recall that with gravity (a conservative, central force) there are some quantities which will not change in time, even as the position and velocities of the bodies seemingly change in a random fashion: energy, angular momentum, and linear momentum are conserved.

$$
\begin{aligned}
E &= \frac{1}{2}\left(m_1v_1^2+m_2v_2^2+m_3v_3^2\right) - \frac{Gm_1m_2}{|\mathbf{r}_2-\mathbf{r}_1|} - \frac{Gm_1m_3}{|\mathbf{r}_3-\mathbf{r}_1|} - \frac{Gm_3m_2}{|\mathbf{r}_2-\mathbf{r}_3|} \\
&= \frac{ML^2}{T^2}\left[\frac{1}{2}\left(m'_1v'^2_1+m'_2v'^2_2+m'_3v'^2_3\right) - \frac{GT^2M}{L^3}\left(\frac{m'_1m'_2}{|\mathbf{r}'_2-\mathbf{r}'_1|}+\frac{m'_1m'_3}{|\mathbf{r}'_3-\mathbf{r}'_1|}-\frac{m'_3m'_2}{|\mathbf{r}'_2-\mathbf{r}'_3|}\right)\right] \\
E' &= \frac{1}{2}\left(m'_1v'^2_1+m'_2v'^2_2+m'_3v'^2_3\right) - \left(\frac{m'_1m'_2}{|\mathbf{r}'_2-\mathbf{r}'_1|}+\frac{m'_1m'_3}{|\mathbf{r}'_3-\mathbf{r}'_1|}+\frac{m'_3m'_2}{|\mathbf{r}'_2-\mathbf{r}'_3|}\right) \\
\mathbf{L} &= m_1\mathbf{r}_1\times\mathbf{v}_1+m_2\mathbf{r}_2\times\mathbf{v}_2+m_3\mathbf{r}_3\times\mathbf{v}_3 \\
L_z &= \frac{ML}{T}\left[m'_1(x'_1v'_{1y}-y'_1v'_{1x})+m'_2(x'_2v'_{2y}-y'_2v'_{2x})+m'_3(x'_3v'_{3y}-y'_3v'_{3x})\right] \\
L'_z &= m'_1(x'_1v'_{1y}-y'_1v'_{1x})+m'_2(x'_2v'_{2y}-y'_2v'_{2x})+m'_3(x'_3v'_{3y}-y'_3v'_{3x}) \\
\mathbf{P} &= m_1\mathbf{v}_1+m_2\mathbf{v}_2+m_3\mathbf{v}_3 \\
&= \frac{ML}{T}\left[m'_1\mathbf{v}'_1+m'_2\mathbf{v}'_2+m'_3\mathbf{v}'_3\right] \\
\mathbf{P}' &= m'_1\mathbf{v}'_1+m'_2\mathbf{v}'_2+m'_3\mathbf{v}'_3
\end{aligned}
$$

Now that we have simple equations, we can drop the primes everywhere as long as we remember that distances, times and masses are in terms of our standard units.

2 Henri Poincaré

The winner of King Oscar's prize was the French mathematician Henri Poincaré (1854–1912). In his famous book *Men of Mathematics*, E. T. Bell calls Poincaré "the last universalist":[1], the last mathematician to work in essentially all areas of mathematics and making important contributions to differential equations and mathematical physics, automorphic functions, topology, combinatorics, and non-Euclidean geometry. Poincaré began his college studies as student of mining engineering. Although he eventually switched to math, his interests remained in what we would call "applied" math, but which he called "true mathematics—that which serves some useful purpose". Poincaré was not sympathetic to the rise of rigor in mathematics; this rise of rigor over-emphasized the unusual and unuseful, and hid the main path. He objected to the "bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some useful purpose". Oddly enough these bizarre functions end up at center stage in the mathematics he founded: the study of chaos in deterministic differential equations. In writing of Poincaré's thesis, French geometer Gaston Darboux said: "if an accurate idea of the way Poincaré worked is wanted, many points called for corrections or explanations. Poincaré was an intuitionist. Having arrived at the summit he never retraced steps. . . he left to others the pains of mapping the royal roads".

[1] As we will compare Einstein and Poincaré, it is interesting to know how Einstein rated himself in this regard: "The fact that I neglected mathematics to a certain extent had its cause not merely in my stronger interest in the natural sciences than in mathematics but also in following peculiar experience. I saw that mathematics was split up into numerous specialties, each of which could easily absorb the short lifetime granted to us. Consequently, I saw myself in the position of Buridan's ass, which was unable to decide upon any particular bundle of hay" [and hence died of starvation].

3 "Crime" and "Cover-up"

The closing date for entries for King Oscar's Prize was 1888 June 1; King Oscar's 60^{th} birthday was 1889 January 21. (In days before FedEx, one mailed early; Poincaré submitted his manuscript 1888 May 17.) It may seem that nearly eight months is plenty of time to "grade the exam", but the entries were often hundreds of pages of *new* mathematics. Even the most careful mathematicians may fail to notice some pathological possibility, and a gap develops in the chain of reasoning. The judges (Mittag-Leffler and German mathematician Karl Weierstrass) were hard pressed to wade through the material in detail. While one essay stood out (Poincaré's), there were problems. Weierstrass wrote: "As a judge I have been unable to correct some possible errors in Poincaré's work, but I have submitted annotations to the King, asking that *Acta Mathematica* print those annotations in the same issue as Poincaré's work". In fact Weierstrass's report was received *after* the decision had been made and his corrections were never printed.

The long process of editing, typesetting and printing Poincaré's 158 page work took from 1889 April to 1889 November. The back-and-forth editing process forced Poincaré to reconsider his work. On 1889 November 30 a telegraph from Poincaré arrived for Mittag-Leffler. Stop the presses—I've found a critical error. Mittag-Leffler was now is a singularly awkward position. He had sold his king on this problem and sat as a judge and certified this solution. Now Poincaré told him the solution was wrong, and the reverse was probably true, but Poincaré as yet had no proof. In addition, Mittag-Leffler had been forced to publicly defend Poincaré's (incorrect) solution in troublesome debates in the Swedish Academy of Science. (Astronomer and colleague Hugo Gyldén claimed he had already proved and published Poincaré's [false!] result.) Furthermore, it was too late to stop the presses; the volume was done, and preliminary copies had been sent to prominent mathematicians and astronomers. Working feverishly, it took Poincaré about a month to correct his error; his manuscript grew to 270 pages. In today's world in is hard to believe that the gaff would escape notice. But in fact Mittag-Leffler quietly recalled all the volumes containing Poincaré's erroneous proof, and substituted a newly printed volume with the correct proof.[2] (Poincaré paid for the new printing, which cost more than the cash prize he had won.) Of course, a few of Mittag-Leffler friends knew the story, but they had little interest in besmirching Mittag-Leffler or mathematics.

4 "Smoking Gun"

It is quite likely that Poincaré's error would have remained a fluffy bit of mathematical folklore. However, in 1985 Richard McGehee[3] on sabbatical from UMn, paid a visit to the former home of Mittag-Leffler which now houses his papers. There he chanced upon some volumes marked (in Swedish) "The whole edition was destroyed. M.L.". These volumes contained Poincaré first and wrong proof.

5 Computational Physics

The mathematical aspects of the N-body problem deal with proofs. King Oscar's prize requested finding something like Taylor's series describing the position of each body as a function of time. Here we set ourselves with an easier task, but a task with its own pitfalls: we seek to compute (and display) the trajectory of the bodies. Newton gives us the differential equation that describes the trajectory; we must use that differential equation to figure the future positions of the bodies.

[2]Covering up already published work is quite hard. (I have direct experience of this on the *Progressive* case.) It is interesting to compare Mittag-Leffler's success with that of another famous case in physics: the attempt of the Catholic church to re-write the published works of Copernicus and Galileo. Following the condemnation of Galileo, the church sent out a request to every Catholic library and university holding these volumes to cross out various sections and to re-word others. Modern scholars have gone back to these old volumes and found that very few were actually modified. The works remained on the shelf untouched; the order of the pope was mostly ineffective.

[3]McGehee is not a bit player in this story of the N-body problem, but there is little room to tell his story. Suffice it to say that McGehee played a key role in proving the Painlevé conjecture. French mathematician and politician Paul Painlevé presented his conjecture in 1895 as a part of a series of lectures in Stockholm attended and financed by King Oscar.

For example, given the differential equation:

$$\frac{d^2x}{dt^2} = a(x, v)$$

where $a(x, v)$ is some known function of the particle's position x and velocity v, and the particles initial position x_0 and velocity v_0, you seek the future location of the particle. Let x_i be the particles future position and v_i the particle's future velocity at the time: $t_i = i\ \Delta t$. The first time step is easy: for small Δt, the change in the particle's position is given by the velocity and the change in the particle's velocity is given by the acceleration:

$$\begin{aligned} x_1 &= x_0 + v_0 \Delta t \\ v_1 &= v_0 + a(x_0, v_0) \Delta t \end{aligned}$$

Each succeeding step is equally easy:

$$\begin{aligned} x_{i+1} &= x_i + v_i \Delta t \\ v_{i+1} &= v_i + a(x_i, v_i) \Delta t \end{aligned}$$

Now these "equations" are really only approximations; an unknown error is made at each step. However, for small enough time steps the error should be small and we can entertain hopes that the resulting trajectory approximates the true path. (Note that it is generally foolish to seek anything more than an approximate solution to the differential equation, because the equations themselves are very likely only approximations. The key idea is to have some idea of the desired accuracy of the solution, and some way of convincing yourself and others that you've obtained the desired accuracy.) However as the number of steps increases the tedium of repeatedly calculating updated positions increases exponentially. Luckily *Mathematica* has the ability to carryout this "numerical" solution of differential equations automatically.

Lets say, for example, that the acceleration function is something simple like:

$$\frac{d^2x}{dt^2} = -\frac{v^2}{x}$$

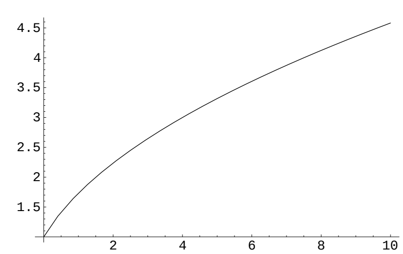
with $x_0 = 1$ and $v_0 = 0$. Then we'd solve this in *Mathematica* as follows:

```
solution=NDSolve[{x''[t]==-x'[t]^2/x[t], x[0]==1,  x'[0]==0}, {x},{t,0,10}]

Out[1]= {{x -> InterpolatingFunction[{{0., 10.}}, <>]}}

ParametricPlot[Evaluate[{t,x[t]} /. First[solution]],{t,0,10}]
```

I suspect you have no idea what the solution should look like, but after you see it you should be able to prove that it is an exact mathematical solution to the differential equation. Change the initial velocity to 1, and again plot the solution. You should find something that looks like:



Furthermore, you should be able to qualitatively explain the result. We start with an initial velocity of 1 (i.e., slope of 1), experience deacceleration, but the amount of deacceleration should decrease as

x grows. If you're acing your differential equation class, you can solve this differential equation (or you might investigate *Mathematica*'s `DSolve` which also knows how to solve it). The solution for these initial conditions is:

$$x(t) = \sqrt{2t + 1}$$

You can see how well *Mathematica* approximated the solution by plotting the difference between the exact solution and the approximate solution:

```
ParametricPlot[Evaluate[{t,x[t]-Sqrt[2 t+1]} /. First[solution]],{t,0,10}]
```

Most people would describe the difference as a small error, but if you're trying to land a probe on a particular spot on Mars from a distance several hundred million miles away, it may look big. *Mathematica* gives us some adjustable parameters to control the error (basically by using smaller time-steps).

- `PrecisionGoal`: limits the percent error in the result. If `PrecisionGoal->`p, then `NDSolve` tries to keep the estimated error in x less than $|x| \times 10^{-p}$. The default is 6. If you care only about absolute error your can: `PrecisionGoal->Infinity`.
- `AccuracyGoal`: limits the absolute error in the result. If `AccuracyGoal->`a, then `NDSolve` tries to keep the estimated error in x less than 10^{-a}. The default is 6. If you care only about percent error your can: `AccuracyGoal->Infinity`.
- `MaxSteps`: In attempting to meet accuracy or precision goals *Mathematica* will decrease the step size, which will result in more steps taken to solve the differential equation over the suggested domain. If more than `MaxSteps` are required, *Mathematica* will discontinue the process and report how far it got in solving the differential equation. Do not be surprised if you must increase this! The default is 1000.
- `WorkingPrecision`: The computer hardware itself limits the accuracy of numbers to this many digits. To change this, buy a new computer (like a Cray). `WorkingPrecision` is 16 digits on these machines.

Increase `PrecisionGoal` and `AccuracyGoal` and see if *Mathematica*'s solution is more accurate:

```
solution=NDSolve[{x''[t]==-x'[t]^2/x[t], x[0]==1,  x'[0]==1}, {x}, {t,0,10},
 PrecisionGoal->7, AccuracyGoal->7]

Out[6]= {{x -> InterpolatingFunction[{{0., 10.}}, <>]}}

ParametricPlot[Evaluate[{t,x[t]-Sqrt[2 t+1]} /. First[solution]],{t,0,10}]
```

Now change the initial velocity to -1... expect problems! You see here an example of a singularity. The solution does not exist beyond $t = \frac{1}{2}$. Plot up the solution as far as it goes. Painlevé's conjecture (discussed in footnote 3) predicted the existence of such singularities in the N-body problem.

6 Meissel's Problem

With three gravitationally interacting bodies, there too many possibilities. As a result most folks (e.g., Poincaré) worked on the simplest possible case: first restrict the motion to a plane, and then eliminate two of the three bodies by assuming that the third body (lets call it the satellite) is so light that it does not affect the motion of the larger two bodies (say the Sun and Earth). If we further assume that the orbit of the Earth around the Sun is circular, we the sort of problem first considered by Lagrange. German mathematician Ernst Meissel (1826–1895) wanted to taste a wider range of possibilities so he invented a particular initial condition, and then explored the future positions of the three bodies. (There is nothing particularly special about Meissel's Problem, it is just a single fairly arbitrary initial state among infinitely more possible cases.) Meissel considered the case of three bodies, with (dimensionless) masses 3, 4, and 5, placed at the corners of a 3–4–5 right triangle with the $m = 3$ mass at the vertex opposite the (dimensionless) length 3 side. At the time, no one really knew what to expect for this complex motion. In 1975 Jörg Waldvogel showed that typically after a close triple encounter one of the bodies becomes unbound and escapes at high speed leaving behind an orbiting binary pair. Your job is to have *Mathematica* `NDSolve` the differential equations of motion and follow the motion of the two bodies. However, there is an pitfall described by Poincaré in his book *Science and Method* (1908):

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still know the [initial] situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say the the phenomenon had been predicted, that is governed by the laws. But it is not always so; it may happen that small differences in initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter... A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance.

For us this pitfall takes a slightly different twist. In a physical problem there are likely to be numerous approximations. We measure initial positions and velocities with limited precision; we don't include all the forces that are acting, only the most obvious; we use Newton's laws even though Einstein taught us that they are only approximately correct. However here mathematician Meissel has exactly specified the initial conditions and the dynamics. There is an exact answer to his problem, but the exact answer doesn't correspond to any real situation and no one expects to find it. Numerical solution offers an approximation, but Poincaré's work on chaos teaches us that if any step is less than perfect, such approximate solutions may increasingly diverge from the actual solution. Thus your problem is: how far can you trust *Mathematica*'s solution? In the above example we could compare the correct solution to *Mathematica*'s solution, and adjust the `AccuracyGoal`, etc., to improve the agreement. Absent any known solution, the usual method is to increase the precision of the numerical solution until the solutions change only a "negligible" amount. If we refine the numerical solution and see no (or a sufficiently small) change in the results, we declare that the numerical solution must be close to the (unknown) exact solution.

7 Homework

1. Write down the six differential equations for the restricted (planer) three body problem. Write down the six equations for the initial positions of the three bodies. Write down the six equations for the initial velocities (0) of the three bodies.

2. Enter the above conditions into *Mathematica*'s `NDSolve` and display the first 10 years of motion. You can display this motion with a command like:

   ```
   In[10]:= ParametricPlot[Evaluate[{{x1[t],y1[t]},{x2[t],y2[t]},{x3[t],y3[t]}} /.
      First[solution]], {t,0,10},AspectRatio->Automatic,
      PlotStyle->{RGBColor[1,0,0],RGBColor[0,1,0],RGBColor[0,0,1]}]
   ```

   Sometimes it is nice to put labels on the plot showing *when* the body is at the particular location:

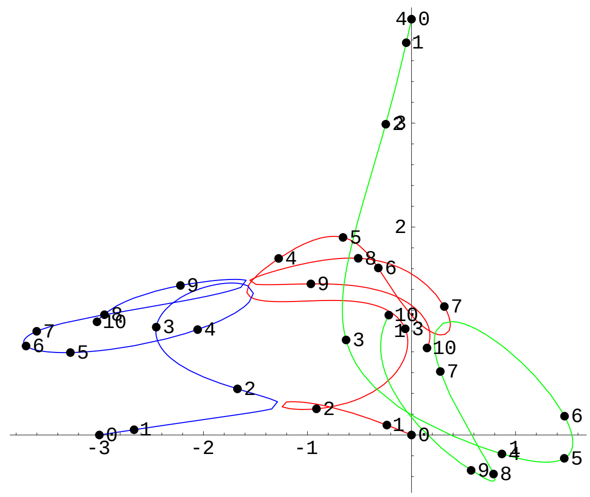
   ```
   <<Graphics/Graphics.m

   Table[{x1[t],y1[t],t},{t,0,10}] /. First[solution]
   LabeledListPlot[%]
   Table[{x2[t],y2[t],t},{t,0,10}] /. First[solution]
   LabeledListPlot[%]
   Table[{x3[t],y3[t],t},{t,0,10}] /. First[solution]
   ```

```
LabeledListPlot[%]

Show[%10,%,%%%,%%%%%]
```

where we assume `%10` is the `ParametricPlot` output.



3. Consider a longer time (say, 30 years). Are the positions accurately reported by *Mathematica* at 30 years? Focus on one body's location at 30 years (e.g., `y2[30]`. Adjust `NDSolve`'s options until you are convinced you have an accurate position for $t = 30$. Record the position of every body at $t = 30$.

4. Repeat the process for $t = 60$. Finally report the longest time for which you believe *Mathematica*'s solution.

5. Use one of the below methods to check the accuracy of your solution described in the previous question.

   (a) One way to check the accuracy of *Mathematica*'s solution is to start the solution from a calculated future point you believe is accurate, and work your way backward in time to the initial values. If the time resversed path takes you back "near" to where you started, that's evidence that the solution is correct. So let's say you want to test if your `solution` is correct at $t = 10$. Save the positions and velocities of the stars using the just complete `NDSolve`:

   ```
   {r1x,r1y}={x1[10],y1[10]} /. First[solution]
   {r2x,r2y}={x2[10],y2[10]} /. First[solution]
   {r3x,r3y}={x3[10],y3[10]} /. First[solution]

   {v1x,v1y}={x1'[10],y1'[10]} /. First[solution]
   {v2x,v2y}={x2'[10],y2'[10]} /. First[solution]
   {v3x,v3y}={x3'[10],y3'[10]} /. First[solution]
   ```

   Now `NDSolve` Newton's equations starting from the above position, but with the reversed velocity. This should move you backward in time:

   ```
   solution2=NDSolve[{x1''[t]==
   ...
   x1'[0]==-v1x,y1'[0]==-v1y,x2'[0]==-v2x,y2'[0]==-v2y,x3'[0]==-v3x,y3'[0]==-v3y,
   x1[0]==r1x,y1[0]==r1y,x2[0]==r2x,y2[0]==r2y,x3[0]==r3x,y3[0]==r3y},
   {x1,y1,x2,y2,x3,y3},{t,0,10}]
   ```

   After those 10 years you should be back to the starting point.

   So using the longest time for which you are convinced of the accuracy of *Mathematica*'s solution, see if "time-reversal" brings you back to the starting position. Report the "inital position" found. How close is it to the actual initial position?

(b) Another way to monitor the accuracy of *Mathematica*'s solution is to have *Mathematica* calculate something whose value you exactly know. About the only things you can exactly know for all future times are the conserved quantities. Plot out one of these conserved quantities as a function of time. For example,

```
e[t_]=.5 m1 (x1'[t]^2+y1'[t]^2)+.5 m2 (x2'[t]^2+y2'[t]^2)+.5 m3 (x3'[t]^2+y3'[t]^2)-
m1 m2/Sqrt[(x1[t]-x2[t])^2+(y1[t]-y2[t])^2] - m2 m3/Sqrt[(x2[t]-x3[t])^2+(y2[t]-y3[t])^2]-
m1 m3/Sqrt[(x1[t]-x3[t])^2+(y1[t]-y3[t])^2]  /. First[solution]
Plot[e[t],{t,0,10}]
```

Is the result "adequately" constant?

6. According to Jörg Waldvogel most choices of initial conditions will lead to an escaping body. Invent your own initial conditions, follow them for as long as you can and report if you see an escaping body.

8 Appendix A

In addition to his (in the end) highly successful work on Newtonian dynamics, Poincaré is also remembered as the man who "could have been Einstein"... who could (should?) have discovered relativity. Poincaré was one of the foremost mathematician of his time; he worked extensively in mathematical physics; he published extensive work on the problems associated with the "luminiferous ether" that immediately pre-staged relativity; he was singularly well prepared to use Riemann geometry (certainly much better prepared than Einstein).

And Poincaré came tantalizingly close to "scooping" Einstein. His opening address to the Paris Congress in 1900, asked *Does the ether really exist?* In 1904 in an address to the International Congress of Arts and Science in St Louis, he pointed out that observers in different frames will have clocks which will "mark what one may call the local time. ... as demanded by the relativity principle the observer cannot know whether he is at rest or in absolute motion." In the same month as Einstein's famous paper, Poincaré stated that "It seems that this impossibility of demonstrating absolute motion is a general law of nature." Meanwhile, other mathematicians made important contributions to relativity, for example, in 1908 Hermann Minkowski (1864–1909) published an important paper on relativity showing the geometrical nature of space-time. (Minkowski told his students at Göttingen, "Einstein's presentation of his deep theory is mathematically awkward—I can say that because he got his mathematical education in Zürich from me".)

Maxwell's laws were relativistically correct (indeed were the foundation that allowed relativity to be discovered), but, of course, Newton's law of gravity was not—so the race was to find the appropriate replacement. Although Poincaré, as an expert in Riemannian geometry, had exactly the right tools, he exactly missed the mark. In his book *Science and Method* (1908) Poincaré wrote:

> If, therefore, we were to discover [experimental evidence of non-Euclidean geometry] we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the laws of optics, and suppose that light is not rigorously propagated in a straight line. It is needless to add that everyone would look upon this [last] solution as the more advantageous. Euclidean geometry, therefore, has nothing to fear from fresh experiments

So how is it that the youthful clerk in the Swiss patent office "scooped" the most famous mathematician of the age in the particular area the mathematician frequently studied? Youth is, of course, one aspect. And Einstein was soon to become the most famous physicist of his age. In addition Einstein was interested in smashing barriers. If we look at the next line of the Einstein quotation that forms the first footnote we have a further hint:

> I saw that mathematics was split up into numerous specialties, each of which could easily absorb the short lifetime granted to us. Consequently, I saw myself in the position of Buridan's ass, which was unable to decide upon any particular bundle of hay. Presumably this

> was because my intuition was not strong enough in the field of mathematics to differentiate clearly the fundamentally important, that which is really basic, from the rest of the more or less dispensable erudition. Also, my interest in the study of nature was no doubt stronger

I conclude that physical intuition is different in kind from mathematical intuition. It is quite odd that we share a language with mathematicians, but nevertheless see the world differently.

9 Appendix B

So far I've had you view moving particles by looking at trajectories. Certainly the best way of understanding the motion would be to see the particle move. *Mathematica* certainly has the ability to make movies, but the results can be confusing because each frame is created before the film is ready to view. First, here is what 60 years of motion looks like:

`http://www.physics.csbsju.edu/orbit/meissel.mov`

Here is how you can create this kind of movie with *Mathematica*:

```
<<Graphics/Animation.m

stars[t_]={RGBColor[1,0,0],Disk[{x1[t],y1[t]},.05],
           RGBColor[0,1,0],Disk[{x2[t],y2[t]},.05],
           RGBColor[0,0,1],Disk[{x3[t],y3[t]},.05]} /. First[solution]

ShowAnimation[Table[ParametricPlot[
     Evaluate[{{x1[t],y1[t]},{x2[t],y2[t]},{x3[t],y3[t]}} /. First[solution]],
     {t,Max[tt-.5,0],tt},AspectRatio->Automatic,PlotRange->{{-4.2,2.2},{-2.2,4.2}},
     PlotStyle->{RGBColor[1,0,0],RGBColor[0,1,0],RGBColor[0,0,1]},
     Epilog->stars[tt]],{tt,.001,10,.1}]]
```

Note that 100 plots will show up; you must activate the animation to make it move!