Vibrations of a String:
“Fourier’s” Series

1 Discussion

The Players:

Leonhard Euler “Oiler” (1707–83) Born in Basel, student of the Bernoullis, Euler answered the call of such notables as Fredrick the Great and Catherine the Great and thus worked at various locations around Europe.

Jean Le Rond d’Alembert (1717–83) Illegitimate son of Chevalier Destouches, D’Alembert had been left on the steps of the chapel Jean Le Rond near Notre-Dame. D’Alembert’s work with partial differential equations is remembered in the name we give to the operator:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2$$

the laplacian minus the time derivatives is the d’alembertian.

Joseph Fourier (1768–1830) Distracted by **1789** Fourier changed his plan to become a Benedictine monk, and instead became a mathematician and friend of Napoleon. Fourier’s use of trigonometric series in his theory of the diffusion of heat (1811 & 1822) comes well after the “Fourier Series” discussed here. As early as 1757 Alexis Claude Clairaut (1713–65) had asserted he could represent *any* function in the form:

$$f(x) = A_0 + 2\sum_{n=1}^{\infty} A_n \cos nx$$

where:

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

However, rigorous proof of the convergence of Fourier Series wasn’t achieved until this century.

The Problem:

Excited by bow or fingers, the plucked string has been an inspiration to “natural philosophers” from Pythagoras to Galileo to modern “string theorists”. Today we immediately write down the partial differential equation that is the wave equation:

$$\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0$$

(where $f(x,t)$ gives the transverse deflection of the string at location x and time t, and v is the wave speed) and proceed to solve it. I will say that d’Alembert first published this equation (in 1746), although others (notably Euler or John Bernoulli) also deserve some credit. D’Alembert was able to show that a general solution to this partial differential equation was the sum of an arbitrarily shaped wave moving to the right and an arbitrarily shaped wave moving to the left:

$$f(x,t) = \psi(x - vt) + \phi(x + vt)$$

Now the plucked string has boundary conditions: for $x < 0$ there is no string and so the function is meaningless there; similarly for $x > L$. Most plucked strings are held in place at the ends so:

$$\begin{aligned} f(0,t) &= 0 \\ f(L,t) &= 0 \end{aligned}$$

are boundary conditions. D'Alembert was able to produce superposition traveling wave solutions and match these boundary conditions by using the trick of considering waves traveling beyond the physical region of the string:

In the physical region, a left moving wave heads towards $x = 0$, as a mirror-image right-moving (dotted) wave in the unphysical region approaches the same boundary. As one wave leaves the physical region, the other emerges, and superposition forces $f(0,t) = 0$. Since the solution is of d'Alembert's form, it is an actual solution to the wave equation that, as was shown, satisfies the boundary conditions.

Within a few months of seeing d'Alembert's paper, Euler solved the same equation by superposition of standing waves. Thus he showed:

$$f(x,t) = A_n \sin\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi n v t}{L}\right) = A_n \sin(k_n x)\ \cos(\omega_n t)$$

also solved the problem and that a general solution would be a superposition of these standing waves:

$$f(x,t) = \sum_{n=1}^{\infty} A_n \sin k_n x\ \cos \omega_n t$$

Its seems that both of these formulations can't be correct (but they are).

***Mathematica's* Solution**

To see that both of these formulations are correct, let's start with a pulse, fourier analyze it, and see if the pulse moves as d'Alembert claimed. Our pulse is mostly zero except a piece that follows a quadratic:

Now according to Euler, the coefficients A_n can be determined from the initial form of the plucked string:

$$f(x,0) = \sum_{n=1}^{\infty} A_n \sin k_n x$$

If we multiply both sides of this equation by $\sin(k_m x)$ and then integrate from 0 to L we can find A_m because almost all of the integrals from the sum are zero:

$$\int_0^L \sin(k_m x)\ \sin(k_n x)\ dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2}L & \text{if } m = n \end{cases}$$

Thus:

$$\int_0^L \sin(k_m x)\ f(x,0)\ dx = \frac{1}{2}\, L A_m$$

In our case $f(x,0)$ is mostly zero, so we can freely reduce the range of integration to where $f(x,0)$ is not zero. At the same time for convenience we set $L = 1$ and $v = 1$:

$$\int_{3/8}^{5/8} \sin(\pi m x)\ \left(x - \frac{3}{8}\right)\left(\frac{5}{8} - x\right)\ dx = \frac{1}{2} A_m$$

Mathematica is happy to do as many of these integrals as we want, but of course, it can't actually do ∞ of them. Luckily the integrals get small and the series converges basically because for large m, the $\sin(k_m x)$ term oscillates many times inside the pulse, and thus averages to near zero. So let's just do the first 50 of these integrations:

`In[1]:= a=Table[0,{n,50}]` ... Create an empty table of A_ns to hold future values

```
In[2]:= Do[a[[n]]=2 Integrate[Sin[Pi n x] (x-3/8)(5/8-x),{x,3/8,5/8}]; Print[n],{n,50}]
```

Do 50 integrations finding A_1 through A_{50}. The `Print` is just to show that something is happening. Now let's form our solution:

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In[3]:= f[x_,t_]=Sum[N[ a[[n]] ] Sin[Pi n x] Cos[Pi n t],{n,1,49,2}]
```

`N` converts the formula for `a[[n]]` into a number. Note that the n-even half of the A_n are zero, so there is no need to add them in.

`Plot[f[x,.1],{x,0,1},PlotRange->{-.02,.02}]` ... Plot out the pulse and see what happens

2 Homework

With pencil and paper, show that both d'Alembert and Euler have valid solutions to the wave equation.

Plot out several snapshots of the above pulses. Produce a hardcopy at one or two times. Describe in words what you see. What is the period of this motion?

Repeat the process with a pulse of your own design.

Turn in a printout showing each step as *Mathematica* solves the problem.

Extra Credit: Design a pulse that looks and moves like the one shown in the discussion of d'Alembert.