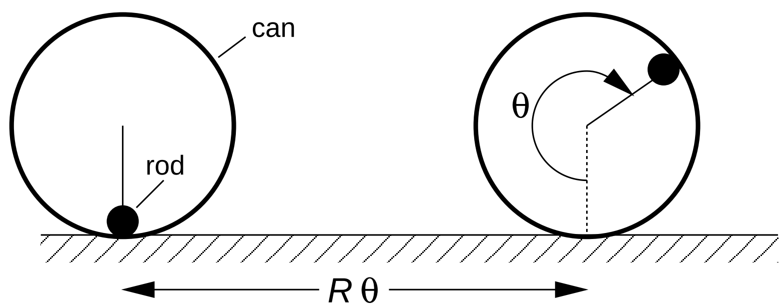
Periodic Hypocycloid Motion
Rod-in-Can Experiment

1 Apparatus

A large coffee can has been turned into a cylinder (radius R, mass M) by removing its top and bottom. On the inside surface of the cylindrical can-edge various solid rods (radius r', mass M') can be attached so that the axis of the rod is a distance $R' \approx R - r'$ from the axis of the can. The combined rod-in-can is free to roll without slipping on a level table. There is a stable equilibrium position with the can rolled so that the rod is directly below the axis of the can. Rolling the rod-in-can away from this stable equilibrium and releasing it results in an oscillation which combines translation and rotation. Technically speaking the path followed by the rod's center of mass is part of a hypocycloid.



2 Experiment

Find the period of the motion $T(M')$ using at least six different rods. Make sure that M' (the rod's mass) spans a large range: from as light as possible (i.e., much lighter than the can) to several times the mass of the can.

Much like a pendulum, the period of oscillation depends slightly on the amplitude of the oscillation. Collect data using "small" amplitude (i.e., amplitudes with less than 30° of rotation; even smaller amplitudes are required when $M' > \frac{1}{2}M$, see below).

3 Theory

Neglecting friction, the motion of the rod-in-can should conserve energy. We begin by finding and then adding together the kinetic energy of the can and the kinetic energy of the rod. We repeatedly use the theorem that the kinetic energy of a system can be calculated by adding the kinetic energy of the system about its (assumed fixed) center of mass and the kinetic energy of the center of mass (assumed to have the entire system's mass).

The center of mass of the can is in the center of its axis. The motion about the center of mass is pure rotation with kinetic energy $\frac{1}{2}I\dot{\theta}^2$ (for a cylinder: $I = MR^2$). The center of mass of the can moves along the tabletop with velocity $R\dot{\theta}$; the contribution to kinetic energy is $\frac{1}{2}Mv^2 = \frac{1}{2}MR^2\dot{\theta}^2$.

Thus the total kinetic energy of the can is:

$$\text{K.E.}_{\text{can}} = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 \approx MR^2\dot{\theta}^2$$

The center of mass of the rod is in the center of its axis. As the rod is rolled around in the can it is rotating around its axis. When the rod-in-can has rolled 180°, the surface of the rod that was below the rod's axis is now above the rod's axis and the *rod* has rotated 180° about its center of mass. Thus the motion of the rod about its center of mass is pure rotation with kinetic energy $\frac{1}{2}I'\dot{\theta}^2$ (for a solid rod $I' = \frac{1}{2}M'r'^2$).

It takes a bit more work to find the kinetic energy of the rod's center of mass on its complex hypocycloid trajectory. We proceed by parameterizing the position of the rod's center of mass in terms of the angle θ, and taking the derivative of the position vector to get the velocity vector.

$$\begin{array}{lll} x: & x = R\theta - R'\sin\theta & \dot{x} = (R - R'\cos\theta)\dot{\theta} \\ y: & y = -R'\cos\theta & \dot{y} = R'\sin\theta\,\dot{\theta} \\ & & \\ v^2: & v^2 = \dot{x}^2 + \dot{y}^2 = \left[(R^2 + R'^2) - 2RR'\cos\theta\right]\dot{\theta}^2 & \end{array}$$

(Our origin is the on the can's axis at the equilibrium position; x is horizonal, y is vertical.)

Thus:

$$\text{K.E.}_{\text{rod}} = \frac{1}{2}M'\left[(R^2 + R'^2) - 2RR'\cos\theta\right]\dot{\theta}^2 + \frac{1}{2}I'\dot{\theta}^2 \approx \frac{1}{2}M'\left[\left(R^2 + R'^2 + \frac{1}{2}r'^2\right) - 2RR'\cos\theta\right]\dot{\theta}^2$$

The total kinetic energy (can and rod) is:

$$\begin{aligned} \text{K.E.}_{\text{total}} &= \frac{1}{2}\dot{\theta}^2\left[M'\left(R^2 + R'^2 + \frac{1}{2}r'^2\right) + 2MR^2 - 2M'RR'\cos\theta\right] \\ &= \frac{1}{2}\dot{\theta}^2\,(2M'RR')\left[1 - \cos\theta + \frac{M'(R - R')^2 + \frac{1}{2}M'r'^2 + 2MR^2}{2M'RR'}\right] \\ &= \dot{\theta}^2\,(M'RR')\left[1 - \cos\theta + \frac{R}{R'}\left(\frac{3}{4}\frac{r'^2}{R^2} + \frac{M}{M'}\right)\right] \\ &= \dot{\theta}^2\,(M'RR')\left[1 - \cos\theta + 2\epsilon\right] \end{aligned}$$

The can's center of mass does not move vertically during the motion and hence does not contribute to the potential energy. The rod's potential energy is:

$$\text{P.E.}_{\text{rod}} = M'gy = -M'gR'\cos\theta$$

The total energy of the system is conserved; it can be conveniently evaluated when the system is at rest at the extreme turning point θ_0. At that point all the energy is in the form of potential energy. Energy conservation now is the equation:

$$\dot{\theta}^2\,(M'RR')\left[1 - \cos\theta + 2\epsilon\right] - M'gR'\cos\theta = -M'gR'\cos\theta_0$$

Solving for $\dot{\theta}^2$ we have:

$$\dot{\theta}^2 = \frac{g}{R}\,\frac{\cos\theta - \cos\theta_0}{1 - \cos\theta + 2\epsilon} = \frac{g}{R}\,\frac{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right) + \epsilon}$$

where we have used the trigonometric identity:

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

It takes $\frac{1}{4}$ period for the rod-in-can to rotate from equilibrium ($\theta = 0$) to maximum ($\theta = \theta_0$):

$$\frac{T}{4} = \int_0^{\theta_0} \frac{d\theta}{\dot\theta} = \sqrt{\frac{R}{g}} \int_0^{\theta_0} \sqrt{\frac{\sin^2\left(\frac{\theta}{2}\right) + \epsilon}{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}} \; d\theta$$

We "simplify" by making the following substitutions:

$$\begin{aligned} T_0 &= 2\pi\sqrt{\frac{R}{g}} \\ k &= \sin\left(\frac{\theta_0}{2}\right) \\ kz &= \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

yielding:

$$T = \frac{4T_0}{\pi} \int_0^1 \frac{\sqrt{k^2z^2+\epsilon}}{\sqrt{1-z^2}\,\sqrt{1-k^2z^2}} \; dz$$

In the small amplitude limit: $\epsilon > k^2 \to 0$ we can Taylor expand the integrand to find:

$$T = 2T_0\sqrt{\epsilon}\left(1 + \frac{1+\epsilon}{16\epsilon}\,\theta_0^2 + \frac{-9+2\epsilon+11\epsilon^2}{3072\epsilon^2}\,\theta_0^4 + \cdots\right)$$

Generally we hope to ignore all but the first term, thus: $T \approx 2T_0\sqrt{\epsilon}$. For $\theta_0 < \sqrt{\epsilon}$ the corrections are generally small. For example, for $\epsilon = .1$ (from, for example, a $M/M' = .2$) and an angular amplitude of $30^\circ \approx 0.524\mathrm{r} > \sqrt{\epsilon}$, the first-term-only version of the period formula is off by 20%, whereas the same system with an angular amplitude of $10^\circ \approx 0.175\mathrm{r} < \sqrt{\epsilon}$ matches the first-term-only period formula to within $2\frac{1}{2}$%. Note that for $M' \gg M$, $\epsilon \to \epsilon_\infty$ and the required "small amplitude" θ_0 becomes quite small ($\sim 5^\circ$) and care is needed if you intend to stay in the small amplitude limit.

In the limit $\epsilon \to \infty$ (for example, $M' \ll M$) the amplitude dependence of the period is exactly the same as for a pendulum. For example with an angular amplitude of 45° the small angle formula is accurate to within 4%.

For $\epsilon = 0$ the integral can be done exactly:

$$T = \frac{2T_0}{\pi}\,\log\left(\frac{1+k}{1-k}\right) \approx \frac{2T_0}{\pi}\,\theta_0\left(1 + \frac{1}{24}\,\theta_0^2 + \frac{1}{384}\,\theta_0^4 + \cdots\right)$$

Note that in this case there is strong (linear) dependence of period on amplitude.

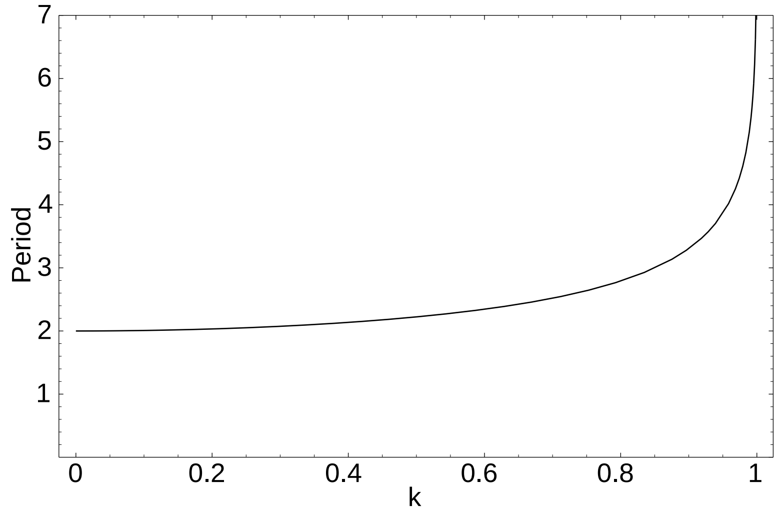


Figure 1: A plot of T/T_0 for $\epsilon = 1$. Note the slight dependence of period on angular amplitude for small amplitude.

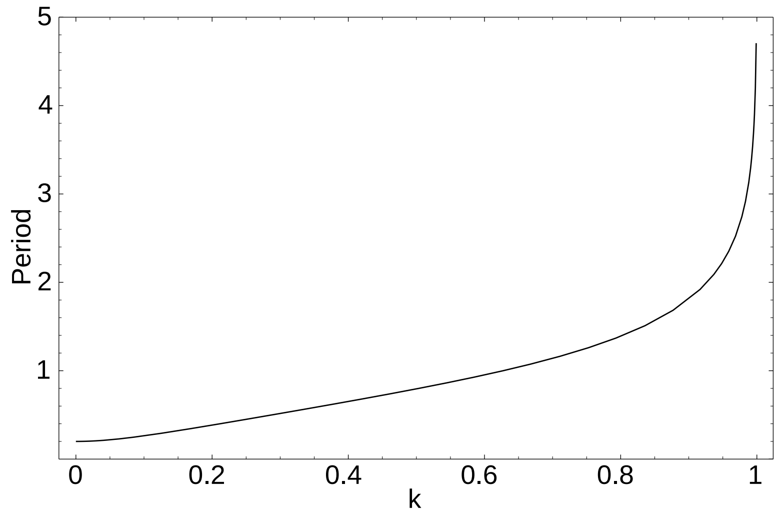


Figure 2: A plot of T/T_0 for $\epsilon = .01$. Note the nearly linear behavior of period with angular amplitude, down the region of $k \sim \sqrt{\epsilon}$, where it finally becomes constant.

4 Results

For small amplitude, we see that

$$T^2 \propto \epsilon = \frac{1}{2}\,\frac{R}{R'}\,\left(\frac{3}{4}\,\frac{r'^2}{R^2} + \frac{M}{M'}\right) \propto A + \frac{B}{M'}$$

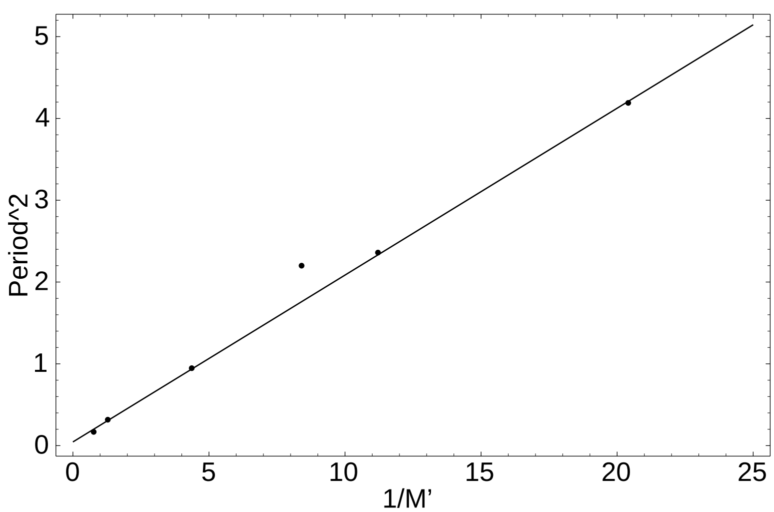
Of particular interest is ϵ_∞, the value of ϵ for infinite M'. The table below records typical results for different coffee cans:

Can	R (cm)	M (g)	ϵ_∞
1 lb	5	115	.02
2 lb	$6\frac{1}{4}$	170	.01
3 lb	$7\frac{1}{2}$	250	.008

For small M' (large ϵ), forces must be large enough to produce oscillation; we find a practical upper bound of $\epsilon \sim 5$.

The following data was collected by a student using the 3 lb coffee can:

M' (g)	T (sec)
49	2.05
89	1.54
119	1.48
229	.97
1,314	.41



The fit line is:

$$T^2(\mathrm{s}^2) = .045 + \frac{.204}{M'(\mathrm{kg})}$$