4. (WKB) Kirkman writes the WKB integral in the form:

$$\int_a^b k(x)dx = \pi(n - \text{something}) \qquad \text{where: } k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar} \tag{1}$$

(a) For each of the below four plots of $V(x)$ report the values for a, b, and "something" if we are considering bound state (or quasi bound state) wavefunctions ψ with an energy, E, of 50.

(b) The lower left potential for $E = 50$ has the "quasi bound state" mentioned above. How does this quasi bound state differ from the other states which are truly bound states? — will tunnel out

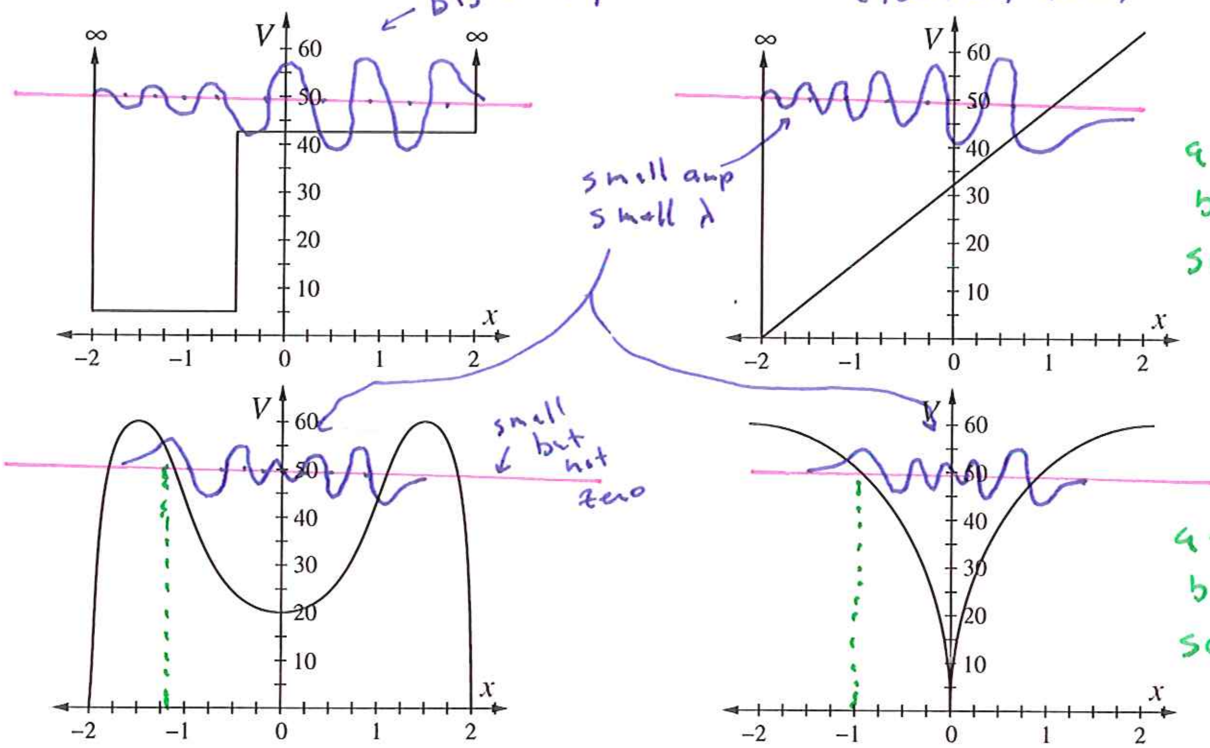
(c) For each of the below potentials, assume that the integral of Eq. 1 produces $n = 10$ for $E = 50$. Sketch the corresponding WKB wavefunction ψ directly on each of the below plots properly displaying changing wavelength & amplitude and behavior near a & b. 9 zeros antisym if possible

← big amp, long λ

(too many zeros)

a = −2
b = 2
some = 0

a = −2
b = 1
some = 1/4



a = −1
b = 1
some = 1/2

a = −1
b = 1
some = 1/2

5. (WKB) Using the WKB approximation, find the formula for the eigenenergies E of a simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

The following integral may be of use:

$$\int_0^A \sqrt{A^2 - x^2}\, dx = \frac{\pi\, A^2}{4}$$

old exam #5

$$-\psi'' = \underbrace{\frac{2m}{\hbar^2}\left[E - \tfrac{1}{2} m\omega^2 x^2\right]}_{k^2} \psi$$

$$\int k\,dx = \frac{\sqrt{2m}}{\hbar} \int \left[E - \tfrac{1}{2} m\omega^2 x^2\right]^{1/2} dx$$

turning point $= A = \sqrt{\dfrac{2E}{m\omega^2}}$

$$= \frac{\sqrt{2m}}{\hbar}\, 2\int_0^A \left[A^2 - x^2\right]^{1/2} \left[\tfrac{1}{2} m\omega^2\right]^{1/2} dx$$

$$= \frac{m\omega}{\hbar}\, 2\int_0^A \left[A^2 - x^2\right]^{1/2} dx = \frac{m\omega}{\hbar}\, 2\, \frac{\pi A^2}{4}$$

$$= \pi \frac{m\omega}{2\hbar} \frac{2E}{m\omega^2} = \pi \frac{E}{\hbar\omega} = \pi\,(n + 1/2)$$

$n = 0, 1, 2, \ldots$

$$\Rightarrow E = \hbar\omega(n + 1/2)$$

WKB: $-\psi'' = k^2 \psi$ $\quad k^2 = E + \dfrac{V_0}{\cosh^2(x)} = E + \dfrac{V_0}{1+u^2} = \dfrac{V_0}{1+u^2} - |E|$ $\quad E = -|E|$

$$\frac{dx}{du} = \frac{1}{\frac{du}{dx}} = \frac{1}{\cosh(x)} = \frac{1}{\sqrt{1+u^2}} \qquad u = \sinh(x)$$

$$\int k\,dx = \int \sqrt{\frac{V_0}{1+u^2} - |E|}\;\frac{1}{\sqrt{1+u^2}}\,du$$

$$= \int \frac{\left((V_0 - |E|) - |E|u^2\right)^{1/2}}{1+u^2}\,du$$

$$= |E|^{1/2} \int \frac{\left[\underbrace{\left(\frac{V_0}{|E|} - 1\right)}_{\text{define as } A^2} - u^2\right]^{1/2}}{1+u^2}\,du$$

$$= |E|^{1/2}\, 2 \int_0^A \frac{\left[A^2 - u^2\right]^{1/2}}{1+u^2}\,du$$

$$= |E|^{1/2}\, 2\, \frac{\pi}{2} \left(\sqrt{\underbrace{1+A^2}_{\frac{V_0}{|E|}}} - 1\right)$$

$$= \pi\left(\sqrt{V_0} - |E|^{1/2}\right)$$

$$= \pi\,(n + 1/2) \qquad n = 0, 1, 2,$$

$$\sqrt{V_0} - (n+1/2) = |E|^{1/2}$$

$$\left(\sqrt{V_0} - (n+1/2)\right)^2 = -E$$

$$V_0 - 2\sqrt{V_0}\,(n+1/2) + (n+1/2)^2$$

same as exact except $V_0 \to V_0 + 1/4$