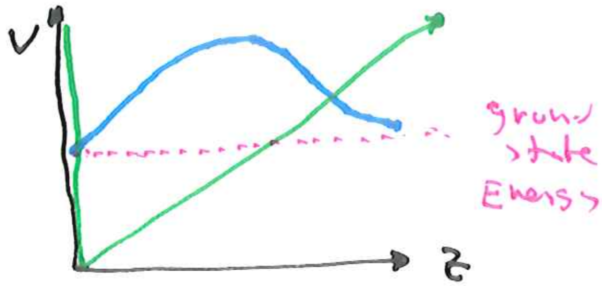
Example: Rayleigh-Ritz - "falling" - $V(z) = \begin{cases} \infty & z<0 \\ mgz & z>0 \end{cases}$



$\psi = z\, e^{-qz}$

$\psi' = (1-qz)\, e^{-qz}$

Dimensionless coordinates:

$$\left(-\frac{\hbar^2}{2m}\partial_z^2 + mgz\right)\psi = E\psi$$

A ← units $E\cdot L^2$ B units $\frac{E}{L}$

make length unit $l = (A/B)^{1/3}$

energy unit $e = (AB^2)^{1/3}$

$z = z' l$

$E = E' e$

$$\left(-\frac{\hbar^2}{2ml^2}\partial_{z'}^2 + mgl z'\right)\psi = E' e\,\psi$$

$B\,(A/B)^{1/3} = (AB^2)^{1/3} = e$

$$\frac{A}{(A/B)^{2/3}} = (AB^2)^{1/3} = e$$

use: $\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$

$$\left(-\partial_{z'}^2 + z'\right)\psi = E'\psi$$

$(1 - 2qz + q^2z^2)$

$\frac{0!}{2q}$ $\frac{2q\, 1!}{(2q)^2}$ $\frac{q^2\, 2!}{(2q)^3}$

$$KE = \langle\psi'|\psi'\rangle = \int_0^\infty (1-qz)^2 e^{-2qz} dz$$

$$= \frac{1}{4q}$$

$$PE = \langle\psi|z|\psi\rangle = \int_0^\infty z^3 e^{-2qz} dz = \frac{3!}{(2q)^4} = \frac{3}{8q^4}$$

$$N = \langle\psi|\psi\rangle = \int_0^\infty z^2 e^{-2qz} dz = \frac{2!}{(2q)^3} = \frac{1}{4q^3}$$

$$\frac{KE+PE}{N} = q^2 + \frac{3}{2q} = 2.476$$

(Note exact answer is 2.338)

$$[\;]' = 0 = 2q - \frac{3}{2q^2} \Rightarrow q^3 = \frac{3}{4}$$

WKB: (Wentzel, Kramers, Brillouin) $-\psi'' = \underbrace{k^2(x)}_{\frac{2m}{\hbar^2}(E-V(x))}\psi$

Remark: $\hbar k(x) = \sqrt{2m\underbrace{(E-V)}_{KE=\frac{p^2}{2m}}} = p$

Remark 2: in classically disallowed $V > E \Rightarrow k(x) = i\underset{\text{Kappa}}{K}$

For an oscillating ψ try $\psi = e^{iS(x)}$ (WOLOG) where expect

$$S(x_0+\Delta x) = S(x_0) + S'\Delta x = \text{const} + 2\pi\frac{\Delta x}{\lambda}$$

Here we have a "changing wavelength" so $\lambda(x)$

$\psi = e^{iS}$ assume "small" so $S' = \pm k \Rightarrow S = \int^x k(x)dx$

$\psi' = iS'e^{iS}$

$\psi'' = [(iS')^2 + iS'']e^{iS} = -k^2(x)\psi$ ← assuming this is "small"

WOLOG: $S = \int^x k(x)dx + \varepsilon$

$S' = k(x) + \varepsilon'$

$S'' = k' + \varepsilon''$

$$[iS'' - S'^2] = [i(k'+\varepsilon'') - (k+\varepsilon')^2] = -k^2$$

$$ik' - 2k\varepsilon' \underbrace{-\varepsilon'^2 + i\varepsilon''}_{\text{neglect}} = 0$$

$$\Rightarrow \varepsilon' = i\frac{k'}{2k} \Rightarrow \varepsilon = \frac{i}{2}\ln(k) + \text{const}$$

$$= i\ln(\sqrt{k}) + \text{const}$$

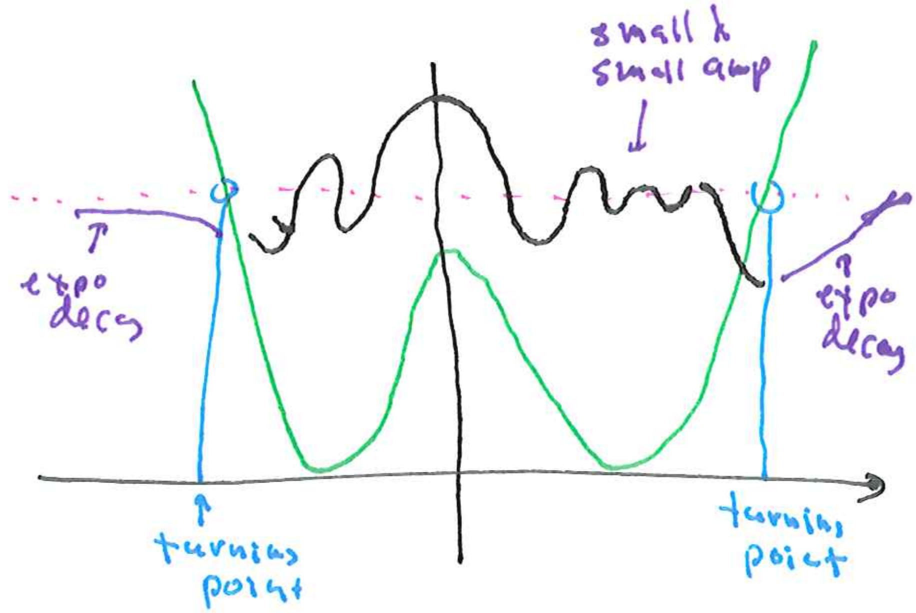
$$\Rightarrow \psi = e^{iS} = e^{i\int k\,dx}e^{-\ln(\sqrt{k})} = \underbrace{\frac{1}{\sqrt{k}}}_{\substack{\text{amplitude large} \\ \text{as } k\to 0 \text{ (turning pt)}}} e^{i\int k\,dx}$$

$k(x) = \frac{2\pi}{\lambda(x)}$

Remark: if in classically disallowed $k = iK \Rightarrow$

$$\psi = \frac{1}{\sqrt{K}}e^{\pm\int K\,dx}$$ ← expo growth/decay

Qualitative wavefunction – Big amplitude where particle slow.
When slow $\lambda = \frac{h}{p}$ Big λ. Expo decay into disallowed region



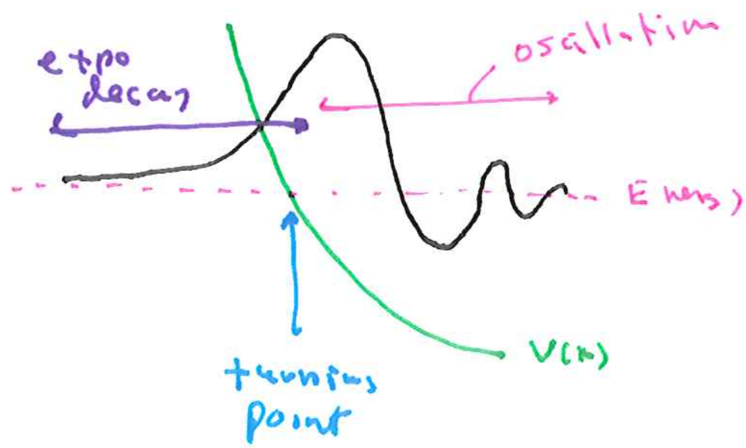
Basic Idea (Bohr)
Fit an integer number of half wavelengths between turning points

$$\Rightarrow \int_{tp_1}^{tp_2} \underbrace{k(x)\,dx}_{k\,dx \,=\, 2\pi \underbrace{\frac{dx}{\lambda}}_{\text{fraction of wavelength}}} = n\pi$$

$$\text{so } \int_{tp}^{tp} k(x)\,dx = 2\pi \underbrace{(\text{total fraction of wavelength})}_{\text{make} = \frac{\text{integer}}{2}}$$

Update Idea – wave function not actually zero at turning point [ie doesn't start as zero has already risen from expo decay]

unless $V = \infty$



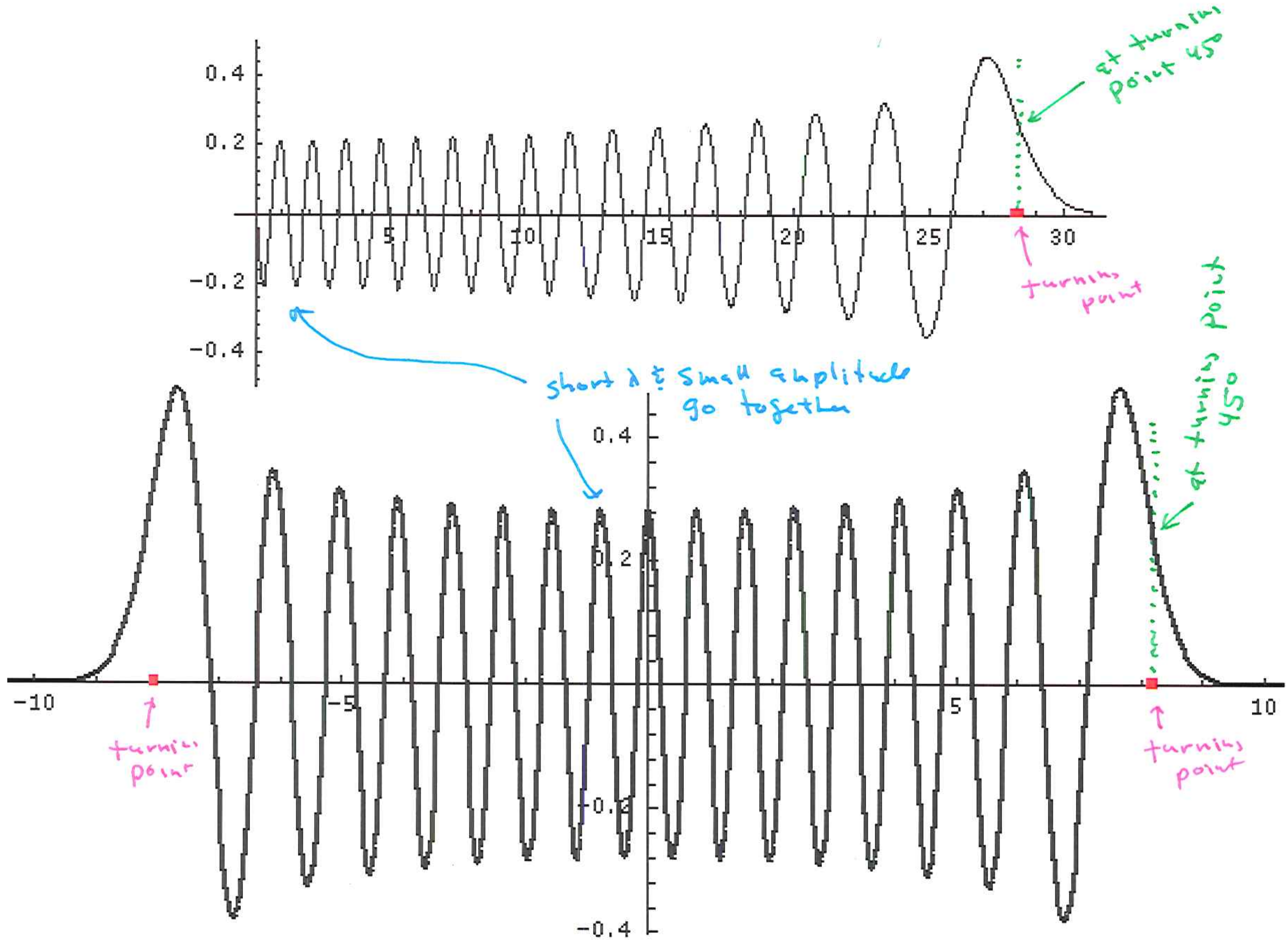
Turns out at turning point its already at $45° = \frac{1}{8}$ wave
→ with turning points at both ends $\frac{1}{4}$ wave missing

⇒ seek integer # half waves <u>minus</u> $\frac{1}{4}$ wave

$$\Rightarrow \int k(x)\,dx = \pi\left(n - \frac{1}{2}\right)$$

half waves

correction: $\frac{1}{4}$ wave



0.4
0.2
-0.2
-0.4
5
10
15
20
25
30
at turning point 45°
turning point
short λ & small amplitude go together
-10
-5
10
turning point
at turning point 45°