QM: far from target (where $V=0$) we've found solutions to TISE: e^{ikz} & $j_\ell(kr)$ & $y_\ell(kr)$

↳ textbook n_ℓ

Hankel $h_\ell^{(1,2)} = j_\ell(kr) \pm i\, y_\ell(kr) \to \frac{1}{kr} e^{\pm i(kr - \frac{\pi}{2}(\ell+1))}$

so general ($r \to \infty$) solution:

scattering amplitude

$$\psi = A\left(e^{ikz} + \frac{f(\theta)}{r} e^{ikr} \right)$$

wave travelling up z axis with flux $|A|^2 \frac{k\hbar}{m}$

radially outward wave with flux $\frac{|A|^2 |f|^2}{r^2} \frac{k\hbar}{m}$

$$\frac{\text{flux of particles into } d\Omega = \frac{|A|^2|f|^2}{r^2}\frac{k\hbar}{m} r^2 d\Omega}{\text{incoming flux} = |A|^2 \frac{k\hbar}{m}}$$

$$= |f|^2 d\Omega$$

so $|f|^2 = \frac{d\sigma}{d\Omega}$ (unit check $[f] = L$ ✓)

Remark: $$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

expresses one azimuthally symmetric solution to TISE in terms of a linear combination of a complete set.

How to calculate $f(\theta)$ given $V(r)$:

start with TISE for radial wavefunction given E & l

$$\left[-\frac{\hbar^2}{2m}\left(\partial_r^2+\frac{2}{r}\partial_r\right)+\frac{\hbar^2 l(l+1)}{2mr^2}+V(r)\right]R=ER$$

starting at origin with regular behavior ($R \propto r^l$)

integrate diff eq to a distance beyond which $V=0$

At that point (& beyond) R must be a linear combination of $j_l(kr)$ & $y_l(kr)$

for $r>R$: $\quad R(r) = A_l j_l(kr) + B_l y_l(kr)$

$$= \sqrt{A_l^2+B_l^2}\,\underbrace{(\cos\delta_l\, j_l - \sin\delta_l\, y_l)}_{\tilde{\varphi}_l}$$

where $\tan\delta_l = \frac{-B_l}{A_l}$ is now determined.

[we have assumed only that R is real as it started

Real $\propto r^l \Rightarrow$ zero net flux: incoming = outgoing]

write $\tilde{\varphi}_l$ in terms of hankel & partially undo

$$\tilde{\varphi}_l = \cos\delta_l\left[\frac{h^{(1)}+h^{(2)}}{2}\right]-\sin\delta_l\left[\frac{h^{(1)}-h^{(2)}}{2i}\right]$$

$$=\frac{1}{2}\left[e^{i\delta_l}h^{(1)}+e^{-i\delta_l}h^{(2)}\right]$$

$$=e^{-i\delta_l}\underbrace{\frac{1}{2}\left[h^{(2)}+e^{2i\delta_l}h^{(1)}\right]}_{\varphi_l \,=\, j_l(kr)+\frac{(e^{2i\delta_l}-1)}{2}h^{(1)}(kr)}$$

Note: for $r<R$
φ_l can be related to original R

Form: $\Psi = \sum (2l+1) i^l \varphi_l P_l(\cos\theta)$

if $r>R$ $\quad = e^{ikz} + \sum (2l+1) i^l \frac{(e^{2i\delta_l}-1)}{2} h^{(1)} P_l(\cos\theta)$

$$h^{(1)} \to \frac{1}{kr}\frac{e^{ikr}}{i^{l+1}}$$

$$= e^{ikz} + \underbrace{\sum \frac{(2l+1)}{k}\frac{(e^{2i\delta_l}-1)}{2i}P_l}_{f(\theta)}\frac{e^{ikr}}{r}$$

$$f(\theta) = \sum \frac{(2\ell+1)}{k} \left(\frac{e^{2i\delta_\ell}-1}{2i}\right) P_\ell(\cos\theta)$$

$f_\ell = e^{i\delta_\ell} \sin\delta_\ell$

$$|f|^2 = \sum_{\ell,\ell'} \frac{(2\ell+1)(2\ell'+1)}{k^2} f_\ell f_{\ell'}^* \underbrace{P_\ell(\cos\theta) P_{\ell'}(\cos\theta)}_{\int d\Omega = \frac{4\pi}{2\ell+1}\delta_{\ell'\ell}}$$

$$\sigma = \int |f|^2 d\Omega = \sum \frac{(2\ell+1)4\pi}{k^2} |f_\ell|^2$$

$\uparrow \sin^2\delta_\ell$

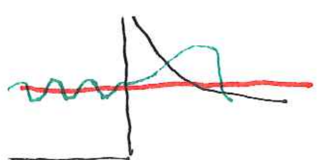
so $\delta_\ell = n\pi \rightarrow$ contributes zero to σ

$\delta_\ell = n\pi + \frac{\pi}{2} \rightarrow$ largest possible contribution.

Note: if all but one $\delta_\ell = n\pi \Rightarrow |f|^2 \propto P_\ell^2(\cos\theta)$

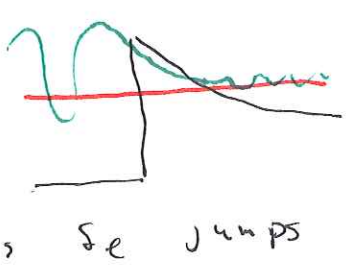
Note: for a repulsive potential at low energy, V lies in classically disallowed region $\rightarrow$ not much effect on $R \approx j_\ell(kr) \Rightarrow \delta_\ell = 0$ (or $n\pi$)

Note: for attractive potential with bound states typical R growing expo from classically disallowed - much as if no attractive potential



so solution $\approx j_\ell(kr) \Rightarrow \delta_\ell = n\pi$

But there is a classically allowed region - there will be oscillations unlike $j_\ell(kr)$. At just the right energy internal solution will connect to exactly expo decay



these will look like quasi bound states

The range of energy that connects to exact expo decay is small

as δ_ℓ jumps from $n\pi$ to $n\pi$ $\rightarrow$ it will briefly hit $n\pi \pm \pi/2$ and make a big contribution to σ

For small k the δ start "at zero" $= n\pi$
The first l to experience the potential is $l=0$, so for δ_0 we make a Taylor series expansion:

$$\delta_0 = -ak + n\pi$$ [note units of $a = L$]

scattering length.

Then $$\underbrace{f(\theta)}_{\text{just } l=0} \approx \frac{1}{k} \underbrace{\frac{e^{2i\delta_0}-1}{2i}}_{\approx \frac{2i\delta_0}{2i} = -ak} \overbrace{P_0(\cos\theta)}^{1} = -a$$

so: isotropic with total cross-section $4\pi a^2$

[note: in the case of hard sphere $R = a$]

In general there is no relation between a & physical size of potential

We can continue the Taylor series expansion of δ_0 but it is usually done in terms of $\cot\delta_0$

[note since δ_0 "starts" at $n\pi$ $\cot\delta_0$ starts at ∞]

$$\cot\delta_0 = \frac{-1}{ka} + \frac{1}{2} r_0 k$$

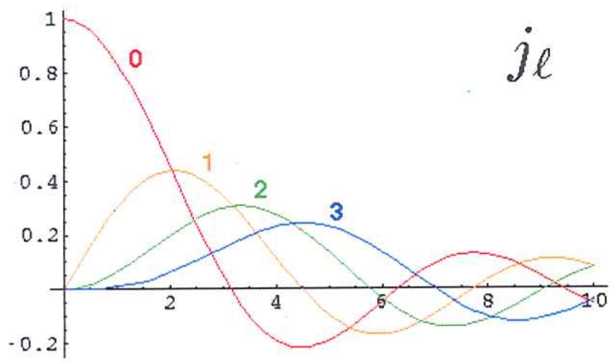
"effective range"

$$\Rightarrow \sigma_0 = \frac{4\pi a^2}{\left[1 - \frac{1}{2} a r_0 k^2\right]^2 + a^2k^2}$$

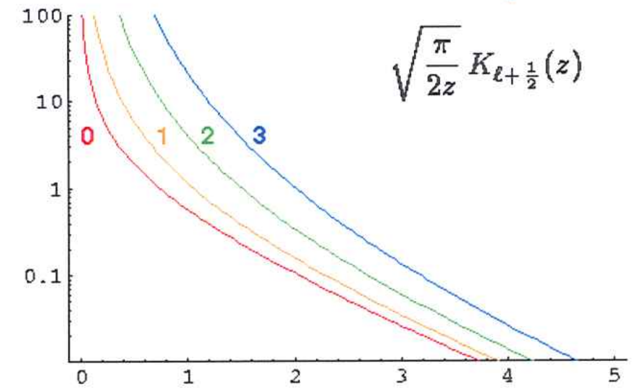
Classical turning pt for $V=0$: $\frac{\hbar^2 l(l+1)}{2mr^2} = \frac{\hbar^2 k^2}{2m}$

$$\frac{\overbrace{\sqrt{l(l+1)}}^{\approx l+\frac{1}{2}}}{k} = r$$

so if $l \neq 0$ & $k \to 0$; $r \to \infty$ & you miss potential
the larger l, the larger k required to bring r into range of U



$$j_0(z) = \frac{\sin z}{z}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$
$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z$$



$$j_l(z)\begin{cases} \xrightarrow[z\to 0]{} & \frac{(\frac{1}{2})!}{(l+\frac{1}{2})!}\left(\frac{z}{2}\right)^l = \frac{1}{(\frac{3}{2})_l}\left(\frac{z}{2}\right)^l \\ \xrightarrow[z\to\infty]{} & \frac{1}{z}\cos\left(z - \frac{\pi}{2}(l+1)\right) \end{cases}$$

$$\left(-\frac{\partial^2}{\partial\rho^2} - \frac{2}{\rho}\frac{\partial}{\partial\rho} + \frac{l(l+1)}{\rho^2}\right)R(\rho) = \pm R(\rho)$$
$$-\left(\frac{1}{\rho^2}\partial_\rho\rho^2\partial_\rho R\right) + \frac{l(l+1)}{\rho^2}R = \pm R$$
$$-(\rho R)'' + \frac{l(l+1)}{\rho^2}\rho R = \pm\rho R$$

$$\sqrt{\frac{\pi}{2z}}\,K_{l+\frac{1}{2}}(z)\begin{cases} \xrightarrow[z\to 0]{} & (\frac{1}{2})!^2\frac{(l-\frac{1}{2})!}{(-\frac{1}{2})!}\left(\frac{2}{z}\right)^{l+1} = \frac{\pi}{4}(\frac{1}{2})_l\left(\frac{2}{z}\right)^{l+1} \\ \xrightarrow[z\to\infty]{} & \frac{\pi}{2z}e^{-z} \end{cases}$$

$$R(\rho) = N j_l(\rho) = N\,\frac{(\rho/2)^l(\frac{1}{2})!}{(l+\frac{1}{2})!}\,{}_0F_1\left(\frac{-}{l+\frac{3}{2}};-\frac{\rho^2}{4}\right)$$

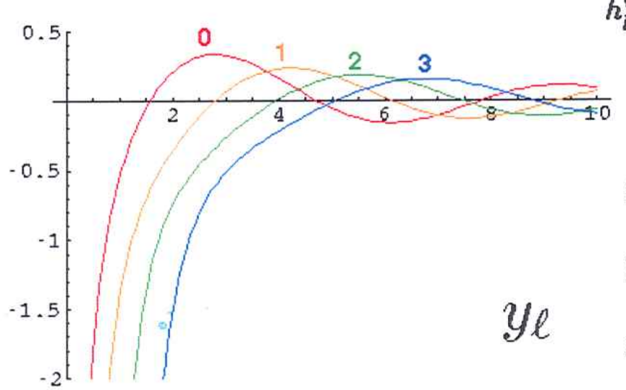
$$\sqrt{\frac{\pi}{2z}}\,K_{0+\frac{1}{2}}(z) = \frac{\pi}{2z}e^{-z}$$
$$\sqrt{\frac{\pi}{2z}}\,K_{1+\frac{1}{2}}(z) = \frac{\pi}{2z}e^{-z}\left(1+\frac{1}{z}\right)$$
$$\sqrt{\frac{\pi}{2z}}\,K_{2+\frac{1}{2}}(z) = \frac{\pi}{2z}e^{-z}\left(1+\frac{3}{z}+\frac{3}{z^2}\right)$$

$$j_l(z) = (\tfrac{1}{2})!\sqrt{\frac{2}{z}}\,J_{l+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,J_{l+\frac{1}{2}}(z)$$

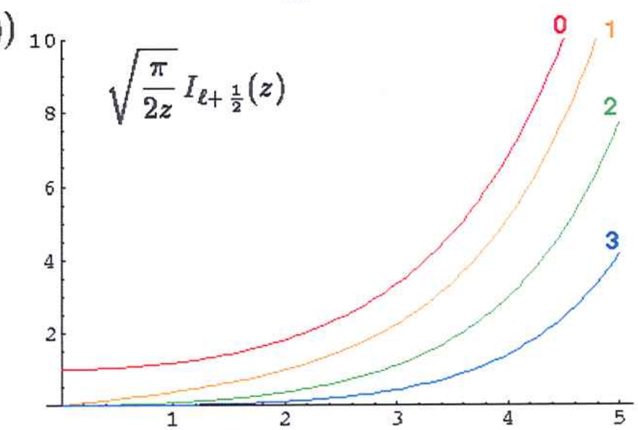
$$y_l(z)\begin{cases} \xrightarrow[z\to 0]{} & -\frac{(l-\frac{1}{2})!}{2(-\frac{1}{2})!}\left(\frac{2}{z}\right)^{l+1} = (-\tfrac{1}{2})_{l+1}\left(\frac{2}{z}\right)^{l+1} \\ \xrightarrow[z\to\infty]{} & \frac{1}{z}\sin\left(z - \frac{\pi}{2}(l+1)\right) \end{cases}$$

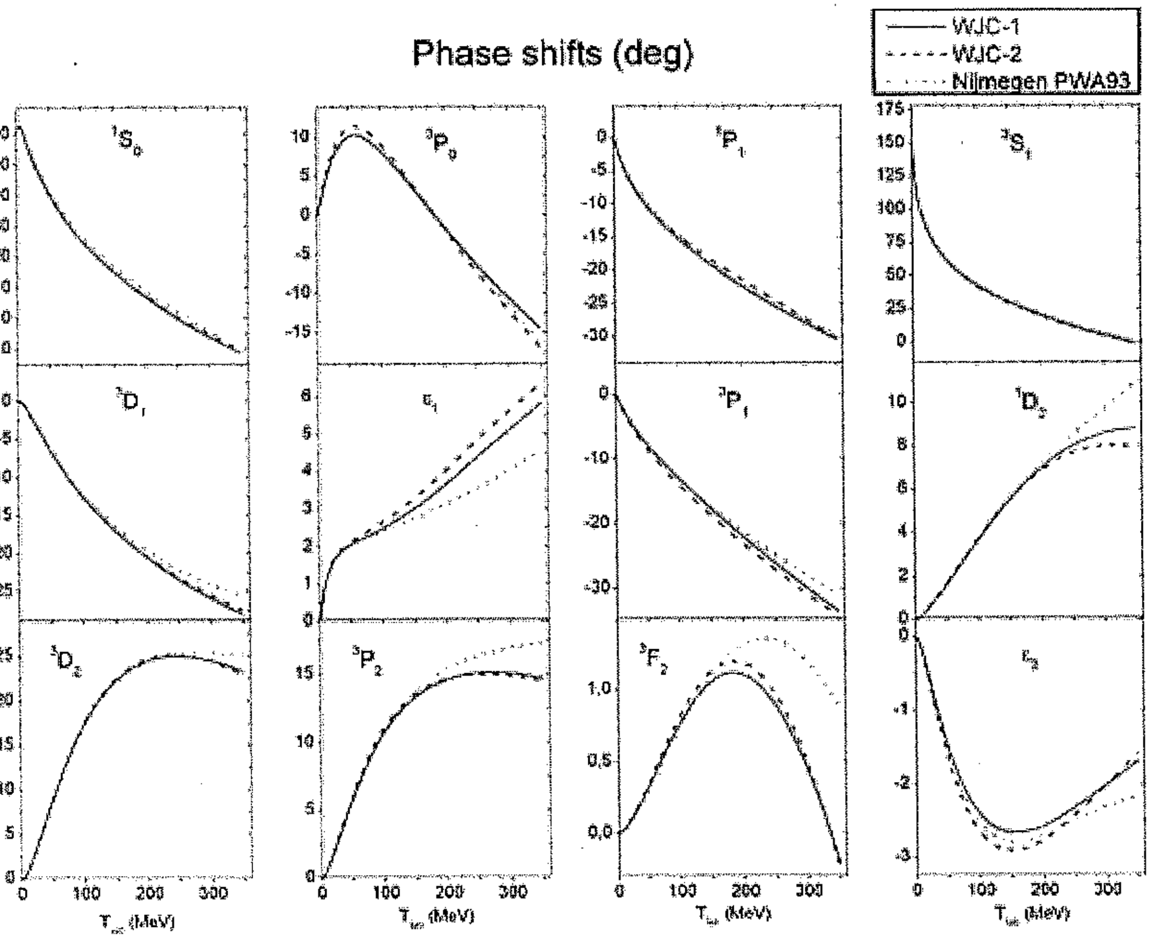
$$\sqrt{\frac{\pi}{2z}}\,I_{l+\frac{1}{2}}(z)\begin{cases} \xrightarrow[z\to 0]{} & \frac{(\frac{1}{2})!}{(l+\frac{1}{2})!}\left(\frac{z}{2}\right)^l = \frac{1}{(\frac{3}{2})_l}\left(\frac{z}{2}\right)^l \\ \xrightarrow[z\to\infty]{} & \frac{1}{2z}e^z \end{cases}$$

$$h_l^{(1,2)}(z) = j_l(z) \pm i y_l(z) \xrightarrow[z\to\infty]{} \frac{1}{z}e^{\pm i\left(z - \frac{\pi}{2}(l+1)\right)}$$



$$y_0(z) = -\frac{\cos z}{z}$$
$$y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$
$$y_2(z) = -\left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z$$





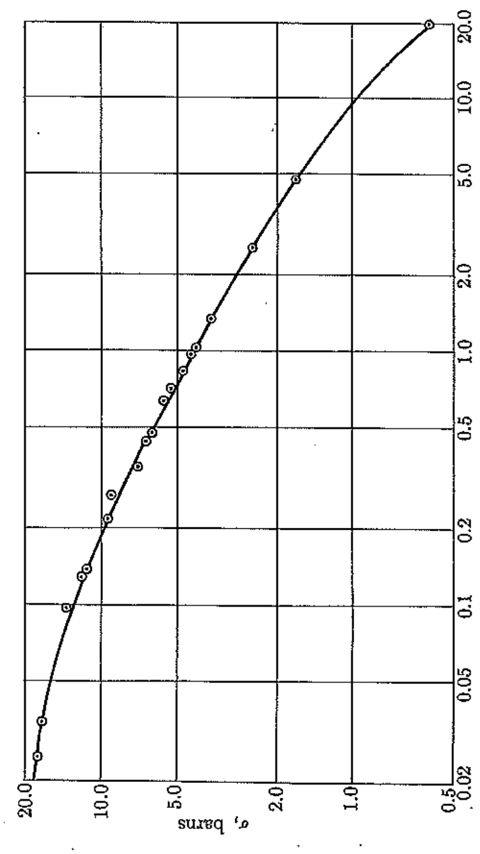


Fig. 3-12. Total cross section for n-p scattering. Theoretical curve based on a_t = 5.38 F, a_s = −23.7 F, r_{0t} = 1.70 F, r_{0s} = 2.40 F. Experimental points from a review paper by R. K. Adair, *Rev. Mod. Phys.* **22**, 249 (1950).