Aim: include $\vec{B}$ ... recall from EM, magnetic dipole PE

$PE = -\vec{\mu}\cdot\vec{B}$ but how does Lorentz force $q\vec{v}\times\vec{B}$ get included in Hamiltonian?

Recall from Mechanics: Lagrangian $L = T - V$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

"canonical momentum" — not exactly same as mv

Hamiltonian: $H = p\dot{q} - L$ ← expressed as function $p\,q$ (remove $\dot{q}$ in favor of p)

$$\frac{\partial H}{\partial p} = \dot{q}$$

$$-\frac{\partial H}{\partial q} = \dot{p}$$

Recall from 200 we expect $m\dot{\vec{v}} = \overbrace{q\vec{v}\times\vec{B}}^{\text{Lorentz Force}}$

Recall from EM: $\vec{B} = \nabla\times\vec{A}$; $\vec{A}$ = vector potential

Gauge invariance $\vec{A}\to\vec{A}+\vec{\nabla}f$ does not change $\vec{B}$

$E = -\nabla\phi - \partial_t\vec{A}$

Begin: express Lorentz Force in terms of $\vec{A}$; $\vec{B}AC - \vec{C}AB$

$$\vec{v}\times\vec{B} = \vec{v}\times(\nabla\times\vec{A}) = v\vec{\nabla}A - v\cdot\nabla\vec{A}$$

connect with dot product

A notation that may not seem to help ...

$$(\vec{v}\times\vec{B})_i = \epsilon_{ijk}v_j(\nabla\times A)_k = \epsilon_{ijk}v_j\epsilon_{klm}\partial_l A_m$$

Now $\epsilon_{ijk}\epsilon_{klm} = \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$$(\vec{v}\times\vec{B})_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_j\partial_l A_m$$

Note Einstein summation notation

$$= \underbrace{v_j\partial_i A_j}_{\text{sum}} - \underbrace{v_j\partial_j}_{\text{sum}} A_i$$

Try Lagrangian $L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}$

voltage PE ← new term

Note: $\vec{P} = \frac{\partial L}{\partial v} = \underbrace{m\vec{v}}_{\text{mechanical momentum}} + \underbrace{q\vec{A}}_{\text{electromagnetic momentum}}$

$\frac{d}{dt}\frac{\partial L}{\partial v}$ will require calculating $\frac{d}{dt}\vec{A}(\vec{r}(t), t)$

additional time variation

location as function of time

$$\frac{d}{dt}\vec{A} = \frac{dx}{dt}\partial_x A + \frac{dy}{dt}\partial_y A + \frac{\partial z}{\partial t}\partial_z A + \partial_t A$$

$$= \vec{v}\cdot\vec{\nabla} A + \partial_t \vec{A}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v} = \frac{d}{dt}(m\vec{v} + q A) = m\dot{\vec{v}} + q(\overbrace{\vec{v}\cdot\vec{\nabla}\vec{A}}^{v_j \partial_j \vec{A}_i} + \partial_t \vec{A})$$

$$= \frac{\partial L}{\partial x} = -q\nabla\phi + q\,\underbrace{\vec{\nabla}(A\cdot v)}_{v_j \partial_i \vec{A}_j}$$

so: $$m\dot{\vec{v}} = -q\nabla\phi - q\partial_t\vec{A} + q(v_j \vec{\partial}_i A_j - v_j \partial_j \vec{A}_i)$$

$$= q\vec{E} + q\,\vec{v\times B}$$

Now form Hamiltonian Note: $m\vec{v} = \vec{p} - q\vec{A}$

$$H = \vec{p}\cdot\vec{v} - L = \vec{p}\cdot\vec{v} - \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi - q\vec{A}\cdot v$$

$\frac{(p - qA)}{m}$ $\qquad$ $\frac{(p-qA)}{m}$

$$= (\vec{p - qA})\cdot\vec{v} - \frac{1}{2m}(p - qA)^2 + q\phi$$

$$= \frac{1}{2m}(\vec{p - qA})^2 + q\phi \leftarrow \text{see its still KE+PE}$$

$p \to \frac{\hbar}{i}\vec{\nabla}$

Note 1: Because of gauge invariance there are many equivalent ways of making a uniform magnetic field. $\vec{B} = B_0\hat{z}$

Eg $$\vec{A} = -\frac{B_0}{2}\vec{r}\times\hat{z} = \frac{B_0}{2}\langle -y, x, 0\rangle$$
or $$\vec{A} = B_0\langle -y, 0, 0\rangle$$

both have $\nabla\cdot A = 0$ a "gauge condition" we sometimes used in EM

in the case of the first example - which is more generally $\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B}$

Note they commute here as A_x doesn't depend on x etc

$$(p - qA)^2 = p^2 \underbrace{- 2q\,A\cdot p}_{\hookrightarrow\, q\,\vec{r\times B}\cdot\vec{p} \,=\, -q\vec{B}\cdot\vec{L}} + q^2 A^2$$

so $$H \to \frac{p^2}{2m} - \underbrace{\frac{q}{2m}\vec{B}\cdot\vec{L}}_{-\frac{qB}{2m}\hbar m} + \underbrace{q^2A^2/2m}_{\text{neglect for the time being}}$$

classical gyromagnetic ratio

$\to$ so a state with L will split into $2\ell+1$ evenly spaced levels "Zeeman Effect"

Example: "cyclotron" charged particle in uniform magnetic field

classical solution: helix with radius $= \frac{mv_\perp}{qB}$

and constant speed in direction of $\vec{B}$ (take as $\hat{z}$)

→ Note these helices can have center at any location in xy plane.

$$H = \frac{p^2}{2m} - \frac{q}{2m}\vec{B}\cdot\vec{L} + \frac{q^2 A^2}{2m} \qquad A^2 \leftarrow \frac{B_0^2}{4}(x^2+y^2)$$

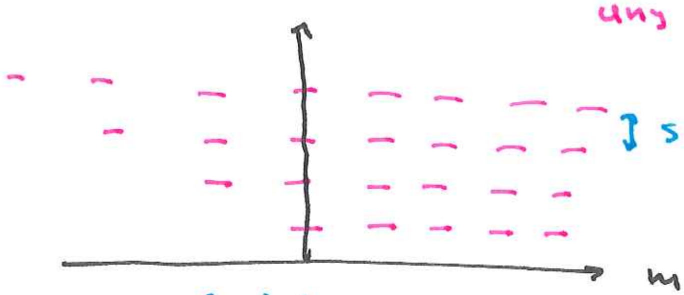
$$= \frac{p_x^2+p_y^2}{2m} + \underbrace{\frac{q^2 B_0^2}{2m4}(x^2+y^2)}_{\text{2d SHO with } \omega = \frac{qB}{2m}} - \frac{qB}{2m}L_z + \frac{p_z^2}{2m}$$

Free in z direction e^{ikz} $E = \frac{\hbar^2 k^2}{2m}$

$$E = \hbar\omega(2n_r + |m| + 1)$$

$$\Rightarrow E = \hbar\omega(2n_r + \underbrace{|m| - m}_{} + 1) + \frac{\hbar^2 k^2}{2m}$$

any $m =$ ~~negative~~ positive integer is degenerate



spacing $= 2\hbar\omega$

B ⊙ Cyclotron classical motion "ω" $= \frac{qB}{m} = 2\omega$ ~~classical~~ QM Larmor

Gauge Transformation if $\vec{A}' = \vec{A} + \vec{\nabla} f$

$\psi' = e^{iqf/\hbar}\psi$

$$(p - qA')\psi' = e^{iqf/\hbar}(p - qA)\psi$$

generalize to f depends on space & time

$\phi' = \phi - \partial_t f$