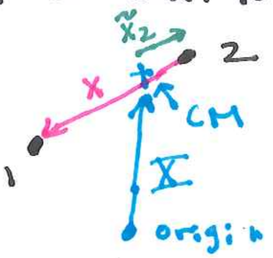
Multiple particles — classical mechanics

CM: $X = \frac{\sum m_i x_i}{M}$ relative coordinate $\tilde{x}_i = x_i - X$

191 thm: total $KE = \underbrace{\frac{1}{2} M V_{cm}^2}_{\text{KE "of" CM}} + \underbrace{\sum \frac{1}{2} m_i \tilde{v}_i^2}_{\text{KE "about" CM, relative KE}}$

Special case: $N=2$: $X = \frac{m_1 x_1 + m_2 x_2}{M}$ $\qquad x_1 = \frac{m_2}{M} x + X$

$x = x_1 - x_2 \qquad x_2 = -\frac{m_1}{M} x + X$



Note: if $m_1 \ll m_2 \approx M$

then $x_2 \approx X$ & $x_1 \approx x + X$

(situation in H atom)

Total $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 \left(\frac{m_2}{M} v + V\right)^2 + \frac{1}{2} m_2 \left(-\frac{m_1}{M} v + V\right)^2$

$= \frac{1}{2} \left(\underbrace{\frac{m_1 m_2^2}{M^2} + \frac{m_2 m_1^2}{M^2}}_{\frac{m_1 m_2}{M^2}(m_2 + m_1) = \frac{m_1 m_2}{M} \equiv \mu \text{ reduced mass}}\right) v^2 + \frac{1}{2} M V^2$

Note: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

$= \frac{1}{2} \mu v^2 + \frac{1}{2} M V^2$

Often: translational invariance — energy does not depend on location of CM $\Rightarrow M V_{cm} = \text{constant}$

$N=2$ QM: $-\frac{\hbar^2}{2}\left(\frac{1}{m_1} \nabla_1^2 + \frac{1}{m_2} \nabla_2^2\right)$

$\partial_{x_1} = \frac{\partial X}{\partial x_1} \partial_X + \frac{\partial x}{\partial x_1} \partial_x = \frac{m_1}{M} \partial_X + \partial_x \longrightarrow \partial_{x_1}^2 = \left(\frac{m_1}{M} \partial_X + \partial_x\right)^2$

$\partial_{x_2} = \qquad = \frac{m_2}{M} \partial_X - \partial_x \longrightarrow \partial_{x_2}^2 = \left(\frac{m_2}{M} \partial_X - \partial_x\right)^2$

seek: $\frac{1}{m_1} \partial_{x_1}^2 + \frac{1}{m_2} \partial_{x_2}^2$

$\hookrightarrow \underbrace{\frac{m_1}{M^2} \partial_X^2 + \frac{m_2}{M^2} \partial_X^2}_{\frac{1}{M} \partial_X^2} + \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \partial_x^2$

So: $-\frac{\hbar^2}{2}\left(\frac{1}{m_1} \nabla_1^2 + \frac{1}{m_2} \nabla_2^2\right) = -\frac{\hbar^2}{2}\left(\frac{1}{M} \nabla_{cm}^2 + \frac{1}{\mu} \nabla_{rel}^2\right)$

Now if V does not depend on $X \Rightarrow H = H(cm) + H(rel)$

So product wave function $\psi = \psi(cm)\, \psi(rel)$

$E_{cm} = \frac{\hbar^2 K^2}{2M}$ & $e^{i K \cdot X}$ $\qquad \hookrightarrow H(rel)\psi = E_{rel} \psi$

Identical particles — wavefunction should just depend on which levels are occupied not which particle is occupying which level. If we swap labels on particles ① & ② it should be exactly the same physically.

For identical particles the operator P that swaps ① ⟷ ② should commute with H: $[H, P] = 0$ & since $P^2 = 1$ the eigenvalues of P are ± 1; these must be conserved.

Claim: there are two types of particles — "Fermions" where swap of any two particles $\Rightarrow P = -1$ and "bosons" where swap of any two particles $\Rightarrow P = +1$

Claim 2: Fermions have "half integer" spin: $\frac{1}{2}, 3/2, 5/2, \ldots$
Bosons have "integer" spin: $0, 1, 2, \ldots$

Fermion examples: e^-, P, quarks ← all spin $1/2$
Δ (spin $3/2$), O^{17} nucleus spin $5/2$

Boson examples: π (spin 0), H^2 nucleus (spin 1), He (0)
special examples — gauge bosons: γ, gluon (spin 1)
Higgs boson (spin 0)

If have product wavefunctions must be $(\psi_a(1)\psi_b(2) \pm \psi_a(2)\psi_b(1))$, with $+$: bose, $-$: Fermi

If have $N=2$ X & x, particle exchange leaves X invariant but $x \to -x$, so if $\psi = \psi_{cm}(X)\,\psi_{rel}(x)$ then ψ_{rel} must be even / odd if particle are bose / fermi

Note: for three or more particles, make symmetric / antisymmetric via determinant ("slater")

$$\psi = \begin{vmatrix} \psi_a(1) & \psi_a(2) & \psi_a(3) \\ \psi_b(1) & \psi_b(2) & \psi_b(3) \\ \psi_c(1) & \psi_c(2) & \psi_c(3) \end{vmatrix}$$

use all positive signs in determinant for symmetric case

Note: particle coordinates include both $\vec{r}$ & spin so can make electron wavefunction using just one spacial state coupled with antisymmetric spin state:

$$\psi = f(\vec{r}_1) f(\vec{r}_2) (\uparrow\downarrow - \downarrow\uparrow)\frac{1}{\sqrt{2}}$$

Thus rule: at most 2 electrons per wavefunction

Note: if electrons have same spin, spacial state must be antisymmetric & hence different

$$\psi = (f(\vec{r}_1) g(r_2) - g(\vec{r}_1) f(\vec{r}_2)) \uparrow\uparrow$$

In more general cases can consider spin states that have mixed symmetry if pair with appropriate spacial state of mixed symmetry.