Recall: we have found orthonormal 'standing' waves on the surface of sphere $\equiv Y_\ell^m(\theta,\phi) = |\ell m\rangle$

"orthonormal" $\Rightarrow \int_0^\pi \int_0^{2\pi} Y_{\ell'}^{m'*} Y_\ell^m \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega} = \delta_{mm'}\delta_{\ell\ell'}$

element for "solid angle"

$r^2 d\Omega$ = surface area

$Y_\ell^m \sim \underbrace{f(\theta)}_{\ell-m \text{ zonal nodes}} e^{im\phi}$

$Y_\ell^\ell \sim (\sin\theta \, e^{i\phi})^\ell \qquad Y_\ell^0 \sim P_\ell(\cos\theta)$ ← Legendre

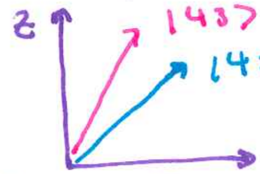
$[L_i, X_j] = i\hbar\epsilon_{ijk}X_k$

$[L_i, \vec{X}\cdot\vec{Y}] = 0$

we used a subset of commutation relations to make operators that changed L_z but left L^2 unchanged

$[L^2, L_\pm] = 0 \qquad L_\pm = L_x \pm iL_y$

$[L_z, L_\pm] = \pm\hbar L_\pm \qquad L^2 = L_\pm L_\mp \mp \hbar L_z + L_z^2$



$L_\pm Y_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}\, Y_\ell^{m\pm1}$

$\left\{ \begin{array}{l} L_z Y_\ell^m = \hbar m Y_\ell^m \\ L^2 Y = \hbar^2\ell(\ell+1)Y \end{array} \right.$

there must be a whole number of states between the largest value of $m = \ell$ & the smallest $m = -\ell$

so $2\ell+1$ = whole number; ℓ = integer or "half integer"

Remark: It is easy to show via $L_\pm$ that

$\langle L_x\rangle = \langle Y_\ell^m | L_x | Y_\ell^m\rangle = 0$ & $\langle L_y\rangle = 0$

so the $\vec{L}$ is usually displayed as a precessing vector on the surface of a cone so, on average, $L_x = 0$ while L_z remains constant at $m\hbar$

see Figure 4.9

$p^2 = -\hbar^2\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{L^2}{r^2}$ so using separation of variables $\psi = R(r)Y_\ell^m(\theta,\phi) \Rightarrow \left[\frac{-\hbar^2}{2m}\underbrace{\left(\partial_r^2 + \frac{2}{r}\partial_r\right)}_{\text{alt: } \frac{1}{r}\partial_r^2(rR)} + \underbrace{\frac{\hbar^2\ell(\ell+1)}{2mr^2}}_{\text{centrifugal potential}} + V\right]R = ER$

if we write $R = \frac{u(r)}{r}$

$\left[-\frac{\hbar^2}{2m}\partial_r^2 + V_{eff}\right]u = Eu$

$V_{eff} = V + \frac{\hbar^2\ell(\ell+1)}{2mr^2}$

now looks like 1d

note: norm $\int \psi^*\psi \, r^2 dr \, d\Omega$ becomes $\int u^* u \, dr \underbrace{\int Y^* Y \, d\Omega}_{\text{orthonormal}}$

Example 5: infinite spherical square well. $V(r) = \begin{cases} 0 & r<a \\ \infty & r>a \end{cases}$

Remark: the 1d square well was easy cuz the diff eq was common & produced familiar functions sin & cos. For higher energy states we just shortened the λ so more oscillations fit in same place. It can't be this easy in 3d as the resulting functions can't be as regular as $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}$ — eg as waves collapse toward origin the same stuff is concentrated in less surface so amplitude must increase — thus functions with decaying oscillations as $r\to\infty$ are the best we can hope for.

$$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\hbar^2 l(l+1)}{2mr^2} + 0\right] R = E R$$

$\times \frac{2m}{\hbar^2}$; define $\frac{2mE}{\hbar^2} \equiv k^2$; define $\rho = kr$

$$\left(-\left(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho\right) + \frac{l(l+1)}{\rho^2}\right) R = 1\, R$$

or $\left(-\partial_\rho^2 + \frac{l(l+1)}{\rho^2}\right) u = u$ where $u = rR$

seek behavior of u or R as $\rho\to\infty \to -u'' = u$

$u = \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\rho)$

seek behavior of R or u as $\rho \to 0$

$R = \frac{1}{r}\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(kr)$

Try power law: $R \sim \rho^a \Rightarrow$

$$-a(a-1)\rho^{a-2} - 2a\rho^{a-2} + l(l+1)\rho^{a-2} = \rho^a$$

$$[l(l+1) - a(a+1)]\rho^0 = \rho^2 \text{ small as } \rho\to 0$$

so $a = l$ or $-(l+1)$

Try power Law: $R = \rho^l \sum a_k \rho^k$

$$\sum -a_k(l+k)(l+k-1)\rho^{l+k-2} - 2a_k(l+k)\rho^{l+k-2} + l(l+1)a_k\rho^{l+k-2} = \sum a_k \rho^{l+k}$$

shift

$$\sum \{l(l+1) - (l+k+2)(l+k+3)\}\, a_{k+2}\, \rho^{l+k}$$

$\hookrightarrow$ factors: $(k+2)(2l+k+3)$

$$a_{k+2} = \frac{-1}{(k+2)(2\ell+k+3)}\, a_k$$ "two term recursion"

$k \to 2i$: $$a_{i+1} = \frac{-1}{(2i+2)(2\ell+2i+3)}\, a_i = \frac{-1/4}{(i+1)(i+\ell+3/2)}\, a_i$$

$$\Rightarrow \sum a_i \rho^{2i} = {}_0F_1\left(\frac{-}{\ell+3/2}; -\rho^2/4\right)$$

hypergeometrics: $${}_1F_1\left(\frac{a}{b}; x\right) = \sum \frac{(a)_n}{(b)_n}\frac{x^n}{n!}$$

shifted factorial or Pochhammer

$$(b)_n = \underbrace{b\,(b+1)\,(b+2)\cdots(b+n-1)}_{n \text{ terms}}$$

$$R \sim \rho^\ell\, {}_0F_1\left(\frac{-}{\ell+3/2}; -\rho^2/4\right) = j_\ell(\rho)$$

Boundary Conditions require $R(a) \sim j_\ell(ka) = 0$

so ka must be a zero of j_ℓ ... call $\beta_{n\ell}$

so $k = \frac{\beta_{n\ell}}{a}$; $E = \frac{\hbar^2}{2ma^2}\beta_{n\ell}^2$

l, n_r	root	E'
0,0	3.1	9.87
1,0	4.5	20.19
2,0	5.8	33.22
0,1	6.3	39.48
3,0	7.0	48.83
1,1	7.7	59.68
4,0	8.2	66.95
2,1	9.1	82.72
5,0	9.4	87.53
0,2	9.4	88.83