$$[\vec{a}\cdot\vec{L}, \vec{X}] = \frac{\hbar}{i}\, \vec{a}\times\vec{X} \qquad \text{where} \quad \vec{X} = \vec{r}, \vec{p}, \vec{L}$$

eg: $\vec{a} = \hat{i} \rightarrow \quad [L_1, \vec{X}] = \frac{\hbar}{i}\, \hat{i}\times\vec{X} = \frac{\hbar}{i}\langle 0, -X_3, X_2\rangle$

so $\quad [X_1, L_1] = 0, \quad [L_1, X_2] = i\hbar X_3, \quad [L_1, X_3] = -i\hbar X_2$

$$[\vec{a}\cdot\vec{L}, r_i] = [\vec{a}\cdot\overrightarrow{r\times p}, r_i] = [(\overrightarrow{a\times r})\cdot\vec{p}, r_i] = \overrightarrow{a\times r}_j\,[p_j, r_i]$$

$$= \frac{\hbar}{i}\,\overrightarrow{a\times r}_j\,\delta_{ij} = \frac{\hbar}{i}\,\overrightarrow{a\times r}_i$$

$$[\vec{a}\cdot\vec{L}, p_i] = [\vec{r}\cdot(\overrightarrow{p\times a}), p_i] = [r_j, p_i]\,\overrightarrow{p\times a}_j = -\frac{\hbar}{i}\delta_{ij}\,\overrightarrow{p\times a}_j$$

$$= \frac{\hbar}{i}\,\overrightarrow{a\times p}_i$$

$$[\vec{a}\cdot\vec{L}, L_i] = \epsilon_{ijk}\,[\vec{a}\cdot\vec{L}, r_j p_k] = \epsilon_{ijk}\left\{[\vec{a}\cdot\vec{L}, r_j]p_k + r_j[\vec{a}\cdot\vec{L}, p_k]\right\}$$

$$= \frac{\hbar}{i}\epsilon_{ijk}\left\{(\overrightarrow{a\times r})_j\, p_k + r_j\,(\overrightarrow{a\times p})_k\right\}$$

$$= \frac{\hbar}{i}\left\{(\overrightarrow{a\times r})\times\vec{p} + \vec{r}\times(\overrightarrow{a\times p})\right\}_i$$

$$= \frac{\hbar}{i}\left\{-\vec{a}(r\cdot p) + \vec{r}\,(a\cdot p) + \vec{a}(r\cdot p) - (r\cdot a)\vec{p}\right\}_i$$

$$= \frac{\hbar}{i}\left\{a\times(\overrightarrow{r\times p})\right\}_i = \frac{\hbar}{i}(\overrightarrow{a\times L})_i$$

NB: Common form $\quad [L_i, X_j] = i\hbar\,\epsilon_{ijk} X_k$

$$[a\cdot L, X\cdot Y] = X_i\,[a\cdot L, Y_i] + [a\cdot L, X_i]\,Y_i$$

$$= \frac{\hbar}{i}\left(X_i\,(\overrightarrow{a\times Y})_i + (\overrightarrow{a\times X})_i Y_i\right)$$

$$= \frac{\hbar}{i}\left(\vec{X}\cdot(\overrightarrow{a\times Y}) + (\overrightarrow{a\times X})\cdot\vec{Y}\right)$$

$$= \frac{\hbar}{i}\left(\epsilon_{ijk}\,X_i a_j Y_k + \underset{\epsilon_{ikj} = -\epsilon_{ijk}}{\underset{\shortparallel}{\epsilon_{kji}}}\, a_j X_i Y_k\right) = 0$$

$L_1 = r_2 p_3 - r_3 p_2 \qquad L_2 = r_3 p_1 - r_1 p_3 \qquad L_3 = r_1 p_2 - r_2 p_1.$

$$L^2 = \sum_{\substack{2,3\\3,1\\1,2}} (r_i p_j - r_j p_i)(r_i p_j - r_j p_i)$$

$$= \sum r_i \overset{\text{free}}{p_j r_i} p_j - r_i \underbrace{p_j r_j}_{= \frac{\hbar}{i} + r_j p_j} p_i - r_j \underbrace{p_i r_i}_{= \frac{\hbar}{i} + r_i p_i} p_j + r_j \overset{\text{free}}{p_i r_j} p_i$$

$$= \sum \big(\underbrace{r_i^2 p_j^2 + r_j^2 p_i^2}_{\text{eg } r_2^2 p_3^2 + r_1^2 p_3^2 = r^2 p_3^2 - r_3^2 p_3^2} - 2 r_i r_j p_i p_j\big) - 2 \frac{\hbar}{i} \vec{r}\cdot\vec{p}$$

$$= r^2 (p_1^2 + p_2^2 + p_3^2) - \sum r_i^2 p_i^2 - 2 \sum r_i r_j p_i p_j - 2 \frac{\hbar}{i} \vec{r}\cdot\vec{p}$$

$$= r^2 p^2 - \sum r_i^2 p_i^2 - 2 \sum_{\text{pairs}} r_i r_j p_i p_j - 2 \frac{\hbar}{i} \vec{r}\cdot\vec{p}$$

$$(\vec{r}\cdot\vec{p})^2 = (r_1 p_1 + r_2 p_2 + r_3 p_3)(r_1 p_1 + r_2 p_2 + r_3 p_3)$$

$$= \sum r_i p_i r_i p_i + 2 \sum_{\text{pairs}} r_i r_j p_i p_j$$

$$= \sum r_i^2 p_i^2 + \frac{\hbar}{i} \vec{r}\cdot\vec{p} + 2 \sum r_i r_j p_i p_j$$

$$\therefore\ L^2 + (\vec{r}\cdot\vec{p})^2 = r^2 p^2 - \frac{\hbar}{i} \vec{r}\cdot\vec{p}$$

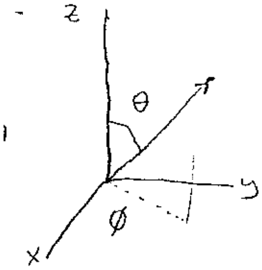
or
$$p^2 = \frac{1}{r^2}\left((\vec{r}\cdot\vec{p})^2 + \frac{\hbar}{i} \vec{r}\cdot\vec{p} + L^2 \right)$$

$$= -\hbar^2 \left(\frac{1}{r^2}\left[r\partial_r r\partial_r + r\partial_r \right] \right) + \frac{L^2}{r^2}$$

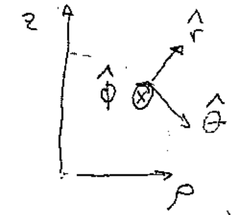
$$= -\hbar^2 \Big(\underbrace{\partial_r^2 + \frac{2}{r}\partial_r}_{\frac{1}{r}\partial_r^2 (r\psi) = \frac{1}{r}\partial_r [\psi + r\psi'] = \frac{2}{r}\psi' + \psi''} \Big) + \frac{L^2}{r^2}$$

✓

$$x = r\sin\theta\cos\phi \qquad\qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\phi \quad \longleftrightarrow \quad \theta = \cos^{-1}\left(\frac{z}{r}\right) \qquad \left(\cos^{-1}(u)\right)' = \frac{-1}{\sqrt{1-u^2}}u'$$
$$z = r\cos\theta \qquad\qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \left(\tan^{-1}(u)\right)' = \frac{1}{1+u^2}u'$$



$$\left.\begin{aligned}\hat{r} = \frac{\vec{r}}{r} &= \langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle \\ \hat{\theta} &= \langle \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\rangle \\ \hat{\phi} &= \langle -\sin\phi, \cos\phi, 0\rangle\end{aligned}\right\} \text{ortho normal}$$



$$\vec{\nabla} r = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}\langle 2x, 2y, 2z\rangle = \frac{\vec{r}}{r} = \hat{r}$$

$$\vec{\nabla}\theta = \frac{-1}{\sin\theta}\left\langle \frac{-zx}{r^3}, \frac{-zy}{r^3}, \frac{-z^2}{r^3}+\frac{1}{r}\right\rangle = \frac{1}{r}\langle\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\rangle = \frac{1}{r}\hat{\theta}$$

$$\vec{\nabla}\phi = \cos^2\phi\left\langle -\frac{y}{x^2}, \frac{1}{x}, 0\right\rangle = \frac{1}{r\sin\theta}\langle -\sin\phi, \cos\phi, 0\rangle = \frac{1}{r\sin\theta}\hat{\phi}$$

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\phi}{\partial x}\partial_\phi$$

$$\vec{\nabla} = (\vec{\nabla} r)\partial_r + (\vec{\nabla}\theta)\partial_\theta + (\vec{\nabla}\phi)\partial_\phi$$

$$\vec{r}\cdot\vec{\nabla} = (\vec{r}\cdot\vec{\nabla} r)\partial_r + 0 + 0 = r\partial_r$$

$$\vec{r}\times\vec{\nabla} = (\vec{r}\times\vec{\nabla} r)\partial_r + (\vec{r}\times\vec{\nabla}\theta)\partial_\theta + (\vec{r}\times\vec{\nabla}\phi)\partial_\phi = \overbrace{(\hat{r}\times\hat{\theta})}^{\hat{\phi}}\partial_\theta + \frac{\overbrace{\hat{r}\times\hat{\phi}}^{-\hat{\theta}}}{\sin\theta}\partial_\phi$$

$$= \left\langle -\sin\phi\,\partial_\theta - \frac{\cos\theta\cos\phi}{\sin\theta}\partial_\phi, \cos\phi\,\partial_\theta - \frac{\cos\theta\sin\phi}{\sin\theta}\partial_\phi, \partial_\phi\right\rangle$$

$$L_\pm = L_x \pm iL_y = \frac{\hbar}{i}\left[\underbrace{(-\sin\phi \pm i\cos\phi)}_{\pm i e^{\pm i\phi}}\partial_\theta - \underbrace{(\cos\phi \pm i\sin\phi)}_{e^{\pm i\phi}}\cot\theta\,\partial_\phi\right]$$

$$= \hbar e^{\pm i\phi}\left[\pm\partial_\theta + i\cot\theta\,\partial_\phi\right]$$

$$L_z = \frac{\hbar}{i}\partial_\phi$$