Position is an example of an operator with a continuous set of eigenvalues

$$\underset{x\text{ operator}}{x}\ \delta(x-a) = \underset{\text{eigenvalue (a real number)}}{a}\ \overbrace{\delta(x-a)}^{\text{eigenfunction } \psi_a}$$

normalization check: $\langle \psi_a | \psi_b \rangle = \int \delta(x-a)\,\delta(x-b)\,dx \overset{?}{=} \delta(a-b)$

Certainly if $a \neq b$ $\int \delta(x-b)\,\delta(x-a)\,dx = \delta(a-b) = 0$

the corresponding "$c(a)$" $= \int \psi_a^* \Psi\,dx = \int \delta(x-a)\Psi\,dx$

$= \Psi(a,t)$ ✓

If we think of x as being one of many possible basis choices, what is the formula for the x operator in other basis? It turns out in the p basis $x = \frac{-\hbar}{i}\partial_p$

The SHO Hamiltonian in the p-basis with then be

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(-\hbar^2\partial_p^2\right) \qquad -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2$$

note that this is very similar to the x version in that both are quadratic + second derivative.

Eg $\quad H = \hbar\omega \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

① Find eigenvectors/values: $\det\begin{pmatrix} a-x & b \\ b & a-x \end{pmatrix} = (a-x)^2 - b^2 = 0$ iff

$(a-x)^2 = b^2$

$(a-x) = \pm b$

$a \mp b = x$

$x = a-b$: $\begin{pmatrix} +b & b \\ b & +b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow b(x+y)=0$
$y = -x$

$x = a+b$: $\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow b(-x+y)=0$
$y = x$

② normalize eigenvectors & check orthogonal

$\vec{v}_1 = N\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \vec{v}_1\cdot\vec{v}_1 = |N|^2 2 \Rightarrow N = \frac{1}{\sqrt{2}}$ $\quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{v}_2 = N\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \vec{v}_2\cdot\vec{v}_2 = |N|^2 2 \Rightarrow N = \frac{1}{\sqrt{2}}$ $\quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{v}_1\cdot\vec{v}_2 = 0$ ✓

③ Expand a given state in terms of eigenvectors

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$ $\quad \alpha_1 = \vec{v}_1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{\sqrt{2}}$

$\alpha_2 = \vec{v}_2\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$

④ write down how this initial state will change with time

$E_1 = \hbar\omega(a-b) \rightarrow \omega_1 = \omega(a-b)$

$E_2 = \hbar\omega(a+b) \rightarrow \omega_2 = \omega(a+b)$

$$\psi(t) = \alpha_1\vec{v}_1 e^{-i\omega_1 t} + \alpha_2\vec{v}_2 e^{-i\omega_2 t} = \begin{pmatrix} -\frac{1}{2}e^{-i\omega_1 t} + \frac{1}{2}e^{-i\omega_2 t} \\ \frac{1}{2}e^{-i\omega_1 t} + \frac{1}{2}e^{-i\omega_2 t} \end{pmatrix}$$

$$= \frac{1}{2}e^{-i\omega a t}\begin{pmatrix} -e^{+i\omega b t} + e^{-i\omega b t} \\ e^{+i\omega b t} + e^{-i\omega b t} \end{pmatrix} = e^{-i\omega a t}\begin{pmatrix} -i\sin(\omega b t) \\ \cos(\omega b t) \end{pmatrix}$$

⑤ check $H\psi = i\hbar\partial_t\psi$

$i\hbar\partial_t\psi$: $\hbar\omega a\, e^{-i\omega a t}\begin{pmatrix} -iS \\ c \end{pmatrix} + i\hbar\omega b\, e^{-i\omega a t}\begin{pmatrix} -ic \\ -s \end{pmatrix}$

$H\psi$: $\hbar\omega e^{-i\omega a t}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} -is \\ c \end{pmatrix} = \hbar\omega e^{-i\omega a t}\begin{pmatrix} -ais + bc \\ -bis + ac \end{pmatrix}$

(2) Lets say we have some other operator $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
we measure S → what do we get & with what prob?

Since the eigenvalues of S are ± 1, every measurement will produce one of those 2 values. To find those probabilities we need to find the eigenvectors of S [easy $\vec{\omega}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{\omega}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$] ← see normalized eigenvectors

Then the prob of measuring $S = +1$ is $|\vec{\omega}_+ \cdot \psi|^2$
prob of measuring $S = -1$ is $(\vec{\omega}_- \cdot \psi)^2$

Now $\psi = e^{-i\omega a t} \begin{pmatrix} -i\sin(\omega b t) \\ \cos(\omega b t) \end{pmatrix}$

$$\text{Prob } S = +1 = (\sin(\omega b t))^2$$
$$S = -1 = (\cos(\omega b t))^2$$

see that the initial state was ω_- so Prob $= 1$ for $S = -1$ but over time the prob that $S = +1$ grows and eventual Prob$(S = +1) = 1$; Prob$(S = -1) = 0$
The probabilities oscillate with time

Pf of CBS "Schwarz" inequality: $|\langle\alpha|\beta\rangle|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$

Note: $\vec{a}\cdot\vec{b} = |a|\,|b|\cos\theta \le |a|\,|b|$

so $|\vec{a}\cdot b|^2 = |a|^2|b|^2\cos^2\theta \le |a|^2|b|^2$

Given vectors $\vec{a}$ & $\vec{b}$ how can we reduce the size of b: $\vec{b} - \underbrace{(\vec{b}\cdot\hat{a})}\hat{a} = \vec{b} - (\vec{b}\cdot\vec{a})\vec{a}/|a|^2$

subtract off all of $\vec{b}$ that is in the $\vec{a}$ direction

$$\left|\,|b\rangle - \frac{\langle a|b\rangle|a\rangle}{\langle a|a\rangle}\right|^2 = \langle b|b\rangle - \frac{\langle b|a\rangle\langle a|b\rangle}{\langle a|a\rangle} - \frac{\langle b|a\rangle\langle a|b\rangle}{\langle a|a\rangle} + \frac{\langle a|b\rangle\langle b|a\rangle\langle a|a\rangle}{\langle a|a\rangle^2} \ge 0$$

$$\langle b|b\rangle - \frac{|\langle b|a\rangle|^2}{\langle a|a\rangle} \ge 0 \rightarrow \langle b|b\rangle\langle a|a\rangle \ge |\langle b|a\rangle|^2$$