Recall dyadic [aka outter product, tensor product] from EM

$$\vec{A}\vec{B} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} (B_x \; B_y \; B_z) = A\tilde{B}$$

Note $\vec{A}\vec{B}\cdot\vec{v} = \vec{A}(B\cdot v)$; $\vec{v}\cdot\vec{A}\vec{B} = \vec{B}(A\cdot v)$

Note: $\hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$ is the identity matrix

$\hat{i}\hat{i}$ is a matrix that, when operating on a vector $\vec{v}$, returns just $v_x\hat{i}$

$\hat{i}\hat{i} + \hat{j}\hat{j}$ returns $v_x\hat{i} + v_y\hat{j}$ [ie throws out v_z]

these are projection operators - note that if they operate twice, the second operation changes nothing

$P^2 = P$

Note: Any orthonormal basis will work just as well as $\hat{i}, \hat{j}, \hat{k}$

Plan: same thing with bra & ket ; define $T = |a\rangle\langle b|$ by $T|c\rangle = |a\rangle\langle b|c\rangle$; Note $T = a b^{+}$ as matrix

Given an orthonormal basis: $\overline{I} = \sum |e_i\rangle\langle e_i|$

To be specific, since this is a basis any vector $\vec{v}$

$\vec{v} = \sum c_n |e_n\rangle$; $\sum |e_i\rangle\langle e_i| \vec{v} = \sum_{in} c_n |e_i\rangle \underbrace{\langle e_i|e_n\rangle}_{\delta_{in}} = \vec{v}$ ✓

$c_n = \langle e_n|v\rangle$

If we use the eigenvectors of a hermitian matrix H as basis then $H = \sum \lambda_i |e_i\rangle\langle e_i|$ ← essentially diagonal form

Basic QM rules: $\forall$ physical quantity (eg angular momentum, energy, position, ...) $\exists$ a corresponding hermitian operator. If you try to measure that physical quantity the only values you will measure are the eigenvalues of the corresponding operator. (Clearly if your measuring instrument lacks resolution, different eigenvalues may be recorded as having the same result — this is frequently the case when macroscopic instruments measure microscopic things) It is admitted a weird idea that the complexities of a real physical instrument (say voltmeter) are somehow represented by a mathematical operator — tough!

Of course repeated measurement of "the same" situation will not produce the same value — thats the intrinsic random of QM. At best we seek the probability of each possible outcome (eigenvalue). that probability is given by $|\langle \psi_a | \Psi \rangle|^2$

normalized eigenfunction that corresponds to the outcome value "a" (eigenvalue)

time dependent wavefunction that describes the system.

Note: since the eigenfunctions of our operator span space "complete"

$$\Psi = \sum c_a |\psi_a\rangle$$

$c_a = \langle \psi_a | \Psi \rangle$ may well depend on time

the set of $\{c_a\}$ are like the coordinates of Ψ. Different operators would provide different basis which would in turn give different coordinates

→ it might be nice to think of Ψ abstractly without any particular choice of basis

→ $\Psi(x,t)$ actually represents a choice of basis — position space — perhaps position basis is not so special

→ the Hamiltonian <u>is</u> a special operator as it determines how things change in time: $H\Psi = i\hbar \partial_t \Psi$

Special case - operators that have a continuum of eigenvalues

discrete: $\Psi = \sum c_n |\psi_n\rangle \xrightarrow[\text{Trick}]{\text{Fourier}} \langle \psi_n | \Psi \rangle = c_n$

continuum: $\Psi = \int c(k) |\psi_k\rangle dk \rightarrow \langle \psi_q | \Psi \rangle = \int c(k) \underbrace{\langle \psi_q | \psi_k \rangle}_{\text{needs to be } \delta(k-q) \text{ to work}} dk$

Remark: $\langle \psi_m | \psi_n \rangle = \delta_{nm}$ ← no units $\rightarrow$ units of $\psi = \frac{1}{\sqrt{L}}$

but if $\langle \psi_q | \psi_k \rangle = \delta(k-q)$ units of $\psi = \frac{1}{\sqrt{L \cdot Q}}$

units of k^{-1}

let units of $k = Q$

seems odd for the same physical quantity to have different units under different circumstances. However all is OK as $|c(k)|^2$ will be a probability density not probability

Momentum is a great example of something with a continuous set of eigenvalues: $\frac{\hbar}{i}\partial_x \underbrace{e^{ipx/\hbar}}_{\text{eigenfunction}} = p e^{ipx/\hbar}$ ← eigenvalue

But mathematics will look cleaner with wave number k

k operator ← $\underbrace{\frac{1}{i}\partial_x}\, e^{ikx} = k e^{ikx}$ ← a fixed real number that is the eigenvalue of this eigenfunction

continuum normalized eigenfunctions of k operator:

$\psi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$

as $\langle \psi_q | \psi_k \rangle = \frac{1}{2\pi}\int e^{i(k-q)x} dx = \delta(k-q)$

Remark: what happened to units? $[k] = \frac{1}{L} = $ "Q"

so $\Psi = \int c(k) \psi_k dk = \frac{1}{\sqrt{2\pi}} \int c(k) e^{ikx} dk$

$c(k) = \langle \psi_k | \Psi \rangle = \frac{1}{\sqrt{2\pi}} \int \Psi(x,t) e^{-ikx} dx$

Remark: is just Fourier Transform.

To convert this to momentum just need to add some $\hbar$

$\psi_p = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$. The resulting $c(p)$ our text calls

$\underbrace{\Phi(p,t)}_{\text{"momentum space wavefunction"}} = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \Psi(x,t) dx$