math vector space over a field of scalars

→ addition that is commutative, associative, ∃ inverses

→ scalar multiplication that is commutative, associative

Note: physics "vector" carries many additional properties to math "vector"

basis vectors $|e_i\rangle$ that span space; dimension

$|\vec{v}\rangle = \sum a_i |e_i\rangle \qquad |\vec{v}\rangle \leftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$ "ket"

Additional property: inner product ("dot product")

$\langle \vec{w} | \vec{v} \rangle = (\langle \vec{v} | \vec{w} \rangle)^*$; $\langle \vec{v} | \vec{v} \rangle \geq 0$ & $= 0$ iff $\vec{v} = \vec{0}$

satisfies linear combinations

orthonormal basis: $\langle e_i | e_j \rangle = \delta_{ij}$

$\langle \vec{w} | = (b_1^*, \cdots, b_N^*)$;

↑ "bra"

$\underbrace{\langle \vec{w} | \vec{v} \rangle}_{\text{bra c ket}} = (b_1^*, \cdots, b_N^*) \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \sum b_i^* a_i$

operator T: $T|e_i\rangle = \sum_j T_{ji} |e_j\rangle$ we use this exact definition so things work out in future

composition (product) of 2 operators [like $g(f(x))$]

$$ST|e_i\rangle = \sum_j T_{ji} S|e_j\rangle = \sum_j T_{ji} \sum_k S_{kj} |e_k\rangle$$

$$= \sum_k \left(\sum_j S_{kj} T_{ji} \right) |e_k\rangle$$

so matrix that represents (ST) is just the matrix product of the matrices that represent S & T

Note: transpose of matrix or vector ~ in Griffiths

hermitian conjugate (adjoint) † (dagger)

see $\langle \vec{w} | \vec{v} \rangle = (|\vec{w}\rangle)^\dagger |\vec{v}\rangle$

$\langle \vec{w} | T\vec{v} \rangle = (T^\dagger |w\rangle)^\dagger |v\rangle = \langle T^\dagger w | v \rangle$

cuz: $\widetilde{(AB)} = \tilde{B}\tilde{A}$ & $T^{\dagger\dagger} = T$

define: Hermitian: $T^\dagger = T$; Unitary $T^\dagger = T^{-1}$

FYI: Symmetric $\tilde{T} = T$; orthogonal $\tilde{T} = T^{-1}$

Eigenvectors & Eigenvalues: $T\vec{v} = \lambda\vec{v}$ "characteristic"

[example: the "principal axes of rotation" were axes for which the angular momentum $\vec{L}$ was in same direction as angular velocity $\vec{\omega}$; i.e., $\vec{L} = \overleftrightarrow{I}\vec{\omega} = \lambda\vec{\omega}$]

if $T\vec{v} = \lambda\vec{v}$ then $(T - \lambda\overleftrightarrow{1})\vec{v} = 0$

unit matrix

so $T - \lambda\overleftrightarrow{1}$ has a null vector & hence has NO inverse.

so $\det(T - \lambda\overleftrightarrow{1}) = 0$

For Hermitian matrices: the eigenvectors are orthogonal (2) & span space; eigenvalues are real (1)

(1): $\langle\vec{v}|T\vec{v}\rangle = \lambda\langle\vec{v}|\vec{v}\rangle = \langle Tv|v\rangle = \lambda^*\langle\vec{v},\vec{v}\rangle$

(2) $\left.\begin{array}{l} T\vec{v} = \lambda\vec{v} \\ T\vec{w} = \mu\vec{w} \end{array}\right\}$ $\langle\vec{w}|T\vec{v}\rangle = \lambda\langle\vec{w}|\vec{v}\rangle$

$\quad\quad\quad\quad\quad\quad \langle T\vec{w}|\vec{v}\rangle = \mu\langle\vec{w}|\vec{v}\rangle$

so, $\underbrace{(\lambda - \mu)}_{\neq 0}\langle\vec{w}|\vec{v}\rangle = 0$

case of "degenerate eigenvalues" via "Gram-Schmidt" process

IF we use eigenvectors as basis, matrix represent T is simple: diagonal

$T|e_i\rangle = \lambda_i|e_i\rangle \Rightarrow T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$

"diagonalization"

How does the matrix that represents an operator change if we select a different basis?

[eg How would the moment of inertia tensor change if we selected a rotated coordinate system?]

Begin by writing the original basis vectors as a linear combination of the new basis (f) - we pick a particular naming for these expansion coeffs:

$$|e_i\rangle = \sum_j S_{ji} |f_j\rangle \xrightarrow[\text{requires}]{} \sum_i (S^{-1})_{ik} |e_i\rangle = |f_k\rangle$$

Matrix representing T in new basis: $(T^f)_{ij} = \langle f_i | T | f_j \rangle$

$$T|f_j\rangle = T \sum_k (S^{-1})_{kj} |e_k\rangle = \sum_k (S^{-1})_{kj} T |e_k\rangle$$

$$= \sum_{k,l} (S^{-1})_{kj} T_{lk} |e_l\rangle$$

$$= \sum_{k,l,n} (S^{-1})_{kj} T_{lk} S_{nl} |f_n\rangle$$

So $\langle f_i | T | f_j \rangle = \sum_{kl} (S^{-1})_{kj} T_{lk} S_{il}$

$$= \sum_{kl} \underbrace{S_{il} T_{lk} (S^{-1})_{kj}}_{\text{matrix } STS^{-1}}$$

Note: if this is a coordinate rotation S will be an orthogonal matrix

if this is a change of orthonormal basis S will be unitary matrix

The determinant & trace of a matrix are invariant under a similarity transform

Eigenvalues also invariant.