Example 4: attractive delta function potential: $V = -\alpha\,\delta(x)$

upshot: 1 "bound state": normalizable & $E<0$

∞ "scattering states": unnormalizable (like e^{ikx}) & $E>0$

function stack:

$\delta(x)$

$\uparrow \frac{d}{dx}$

$\theta(x) = \int_{-\infty}^{x} \delta(y)\,dy$

$\uparrow \frac{d}{dx}$

$\eta(x) = \int_{-\infty}^{x} \theta(y)\,dy$

$\uparrow \frac{d}{dx}$

$V(x) = \int_{-\infty}^{x} \eta(y)\,dy$

$$-\frac{\hbar^2}{2m}\psi'' - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)$$

since there is no $\delta(x)$ on rhs ψ'' must generate a $\delta(x)$ to cancel the one in $V(x)$

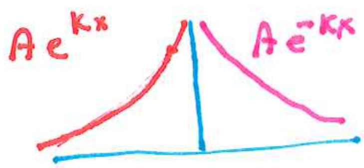
$\Rightarrow$ slope discontinuous @ $x=0$

To find $\int_{0^-}^{0^+}$ diff eq $\Rightarrow$

$$-\frac{\hbar^2}{2m}\left[\psi'(0^+) - \psi'(0^-)\right] - \alpha\,\psi(0) = 0$$

For $x \neq 0$ Diff Eq is easy:

$$\psi'' = \underbrace{-\frac{2mE}{\hbar^2}}_{\text{for } E<0 \text{ this is a positive constant: kappa}^2\ (\kappa^2)}\psi \qquad \text{for } E>0: -k^2$$



$$-\frac{\hbar^2}{2m}\left[-\kappa A - \kappa A\right] = \alpha A \Rightarrow \kappa = \frac{m\alpha}{\hbar^2}$$

$$E = -\frac{\hbar^2\kappa^2}{2m}$$

For scattering states: $\psi = e^{ikx}$ & e^{-ikx}

particular problem incoming beam from left: A

scattered: B

transmitted: C

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{ikx} & x>0 \end{cases}$$

$$\left.\begin{aligned} -\frac{\hbar^2}{2m}\Big[ik\underbrace{C}_{A+B} - (ikA - ikB)\Big] &= \alpha(A+B) \\ &= -\frac{\hbar^2}{m}ikB \end{aligned}\right\} \; B = \frac{im\alpha}{\hbar^2 k}(A+B)$$

$\equiv \beta = \frac{\text{kappa}}{k} = \sqrt{\frac{-E_{BS}}{E}} \sim \frac{\lambda}{\text{BS size}}$

$$\Rightarrow \frac{B}{A} = \frac{i\beta}{1-i\beta} \quad \begin{aligned} &\to 0 \text{ high energy} \\ &\to -1 \text{ low energy} \end{aligned}$$

Reflection $= \left|\frac{B}{A}\right|^2$