3 proofs $\rightarrow$ $\partial_t \rho = -\partial_x J$ (prob conservation)

$$\partial_t \psi^* \psi = \left(\frac{1}{i\hbar} H\psi\right)^* \psi + \psi^* \left(\frac{1}{i\hbar} H\psi\right)$$

$$= \frac{1}{i\hbar}\left[-(\partial_x^2 \psi)^* \psi + \psi^* (\partial_x^2 \psi)\right] \frac{-\hbar^2}{2m}$$

$$= \frac{-\hbar}{2im} \partial_x \left[-(\partial_x \psi)^* \psi + \psi^* (\partial_x \psi)\right]$$

$$= -\partial_x J \quad \text{where } J = \frac{\hbar}{2im}\left[\underbrace{\psi^* \overleftrightarrow{\partial_x} \psi}\right]$$

notation: $\psi^* \partial \psi - (\partial \psi)^* \psi$

Note: Prob particle in $(a,b) = \int_a^b |\psi|^2 dx$

$$\frac{d}{dt}\int_a^b \psi^* \psi \, dx = -\int_a^b \partial_x J \, dx = J(a) - J(b)$$

$J(a)$: current flow in at $a$; $J(b)$: current flow out at $b$

if $b = +\infty$, $a = -\infty$ then $J(b) = 0$, $J(a) = 0$

so total prob is constant

$\rightarrow$ $\frac{d}{dt}\langle x \rangle = \frac{\langle p \rangle}{m}$ ; Lemma $\int (\partial_x \psi^*) \psi \, dx \overset{ibp}{=} -\int \psi^* \partial_x \psi \, dx$

$$\frac{d}{dt}\int_{-\infty}^{\infty} x \psi^* \psi = \int_{-\infty}^{\infty} x \, \partial_t (\psi^* \psi) \, dx = -\int_{-\infty}^{\infty} x \, \partial_x J \, dx \overset{ibp}{=} \int_{-\infty}^{\infty} J \, dx$$

$$= \frac{\hbar}{2im}\int \psi^* \partial_x \psi - (\partial_x \psi)^* \psi \, dx = \frac{\hbar}{im}\int \psi^* \partial_x \psi$$

$$= \frac{1}{m}\int \psi^* \frac{\hbar}{i}\partial_x \psi \, dx = \frac{\langle p \rangle}{m}$$

$\rightarrow$ $\frac{d}{dt}\langle p \rangle = \langle -\partial_x V \rangle$ ; Lemma $\int (p\psi)^* \psi = -\frac{\hbar}{i}\int (\partial_x \psi)^* \psi = \frac{\hbar}{i}\int \psi^* \partial_x \psi \, dx$

$$= \int \psi^* p \psi \, dx$$

$$\frac{d}{dt}\langle p \rangle = \frac{d}{dt}\int \psi^* p \psi \, dx = \int (\partial_t \psi)^* p \psi + \psi^* p \partial_t \psi$$

$$= \frac{1}{i\hbar}\int -(H\psi)^* p \psi + \psi^* p H \psi \, dx \qquad H = \frac{p^2}{2m} + V$$

$\partial_x (V\psi) = (\partial_x V)\psi + V \partial_x \psi$

$$= \frac{1}{i\hbar}\int \underbrace{-\left(\frac{p^2}{2m}\psi\right)^* p \psi + \psi^* p \frac{p^2}{2m}\psi}_{\text{cancel}} - V \psi^* p \psi + \psi^* p V \psi$$

cancel

$$= \frac{1}{i\hbar}\int \psi^* \left(\frac{\hbar}{i}\partial_x V\right) \psi \, dx = -\int \psi^* (\partial_x V) \psi \, dx$$

$$= -\langle \partial_x V \rangle$$

standing waves, stationary states, separation of variables

**Assume** wavefunction can be written as a product of a function of time times a function of space

$\Psi(x,t) = T(t)\,\psi(x)$ ... plug into TDSE

$$i\hbar T'\psi = T\left(\frac{-\hbar^2}{2m}\psi'' + V\psi\right)$$

function only of $t$ and only of $x$

$$\frac{i\hbar T'}{T} = \frac{1}{\psi}\left(\frac{-\hbar^2}{2m}\psi'' + V\psi\right) = E \text{ const}$$

$$\Rightarrow T = e^{-iEt/\hbar} = e^{-i\omega t} \quad \text{if } \omega \equiv E/\hbar$$

$$E\psi = \left(\frac{-\hbar^2}{2m}\psi'' + V\psi\right) = H\psi \quad \text{eigenequation}$$

eigenfunction

eigenvalue

infinite square well: impenetrable wall @ $x=0$ & $x=L$

classical motion: bounce back & forth between walls

"impenetrable" $\Rightarrow \psi = 0$ if $x<0$ or $x>L$

No forces (except $\infty$ force at walls) $\Rightarrow V = 0$

TISE: $E\psi = \left(\frac{-\hbar^2}{2m}\psi'' + 0\cdot\psi\right) \longleftrightarrow \underbrace{\frac{2mE}{\hbar^2}}_{\text{call this } k^2}\psi = -\psi''$

$k^2\psi = -\psi'' \Rightarrow \psi = A\sin(kx) + B\cos(kx)$

require $\psi(0)=0 \Rightarrow B=0$; $\psi(x=L)=0 \Rightarrow kL = n\pi$

$\therefore k = \frac{n\pi}{L}$ & $E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}n^2 \quad n = 1, 2, 3, \ldots$

only possible values of energy

$\psi = A\sin(kx) \qquad A = \frac{1}{\sqrt{\int_0^L \sin^2(kx)\,dx}} = \sqrt{\frac{2}{L}}$

$= \sqrt{\frac{2}{L}}\sin(kx)$ where $k = \frac{n\pi}{L}$

$$\Psi = e^{-i\omega_n t}\sqrt{\frac{2}{L}}\sin(k_n x) \qquad \omega_n = \frac{\hbar\pi^2}{2mL^2}n^2$$

"Fundamental"
"ground state"
$n=1$

even

first excited state
$n=2$

odd

second excited state
$n=3$

even