Physics 346: QM EXAM 3 25 November 2013

Do four problems.

The problems you select should show proficiency in at least three of the four methods we have discussed (RR, WKB, TIPT, TDPT).

1. (RR) Consider a truncated SHO potential:

$$V(x) = \begin{cases} V_0(x^2 - a^2)/a^2 & |x| < a \\ 0 & \text{elsewhere} \end{cases}$$

   (a) The first step in solving this problem is to seek dimensionless coordinates. Clearly we'll want $x' = x/a$, and we can make an energy unit $e = \hbar^2/2ma^2$, so $E' = E/e$. Given the original Hamiltonian:

$$\left[-\frac{\hbar^2}{2m}\,\partial_x^2 + \begin{cases} V_0(x^2 - a^2)/a^2 \\ 0 \end{cases} \right] \psi(x) \;=\; E\psi(x)$$

   write down the Hamiltonian in dimensionless form.

   (b) Consider the trial function:

$$f(x) = xe^{-b|x|}$$

   In order to calculate the trial function's energy expectation value:

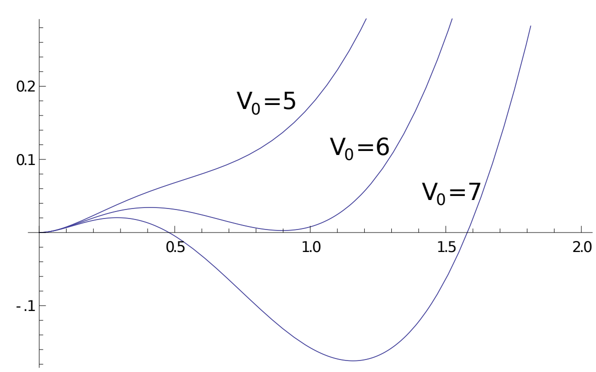
$$E(b) = \frac{\langle f|H|f\rangle}{\langle f|f\rangle}$$

   we must calculate three integrals: $\langle f|f\rangle$, $\langle f|V(x)|f\rangle$, and $\langle f| -\partial_x^2|f\rangle = \langle \partial_x f|\partial_x f\rangle$. Calculate one of these integrals perhaps using:

$$\int_0^\infty x^n\, e^{-\alpha x}\, dx \;=\; n!/\alpha^{n+1}$$

$$\int x^n\, e^{-\alpha x}\, dx \;=\; -\frac{e^{-\alpha x}}{\alpha^{n+1}}\Big[(\alpha x)^n + n\,(\alpha x)^{n-1} + n(n-1)\,(\alpha x)^{n-2} + \cdots + n!\,(\alpha x) + n!\Big]$$

   (c) $E(b)$ is displayed below for $V_0 = 5, 6, 7$. Use this plot to make as precise a statement as possible about the eigen energies (which eigen energies?) of this system for $V_0 = 5, 6, 7$.



2. (TDPT) Consider the symmetric version of a particle-in-a-box with $V(x) = 0$ for $|x| < a$ and $V(x) = \infty$ elsewhere and energy eigenfunctions (sorted by parity)

$$\text{even:} \qquad \psi_n(x) = \frac{1}{\sqrt{a}} \cos(qx) \quad \text{where } n = 1, 3, 5, \ldots$$

$$\text{odd:} \qquad \psi_n(x) = \frac{1}{\sqrt{a}} \sin(qx) \quad \text{where } n = 2, 4, 6, \ldots$$

$$\text{where:} \qquad q = \frac{n\pi}{2a} \quad \text{and} \quad E_n = \frac{(q\hbar)^2}{2m}$$

At $t = 0$ we turn on a small perturbing potential

$$V(x) = \begin{cases} \lambda & |x| < b \\ 0 & \text{elsewhere} \end{cases}$$

where λ is a small constant and $b < a$. (This represents a small 'square' PE bump in the middle of the infinite square well.) The only time dependence in this problem is the turning on of this otherwise constant potential at $t = 0$.

(a) Most of our TDPT work was done under the assumption that terms like V_{aa} were zero (or in the textbook version: $H'_{aa} = 0$). Is this true for this potential? Regardless of how you answer this question answer the below questions using our usual TDPT formulae.

(b) Assume that before the perturbation was applied the particle was in the ground state. Report some states (by their n values) which would become (fractionally) populated during the application of this perturbation according to (usual) first order TDPT.

(c) For one of the states you reported above, report the formula for the probability that the particle is in your reported state after a short time interval T. You need not calculate any integrals, but express any integrals in a form that would make evaluation immediately possible. I.e., for full credit answer with things like:

$$\lambda \int_0^a \cos(\pi x/2a) \cos(3\pi x/2a)\, dx$$

rather than things like:

$$\langle \psi_1 | H' | \psi_3 \rangle$$

3. (TIPT) Consider the same situation as above (symmetric particle-in-a-box with $V(x) = 0$ for $|x| < a$ and $V(x) = \infty$ elsewhere). The eigen energies/functions this unperturbed Hamiltonian are given in the previous problem. Exactly the same perturbing potential

$$V(x) = \begin{cases} \lambda & \text{for: } |x| < b \\ 0 & \text{elsewhere} \end{cases}$$

(where λ is a constant), is present, but in this problem it is/has been continuously present (so the problem is time independent). Write down an expression for the first-order energy shifts of an arbitrary state. Describe the results of this calculation (i.e., which states are affected, how much are they effected, etc.) What states *connect* to the ground state in the second-order energy shift calculation?

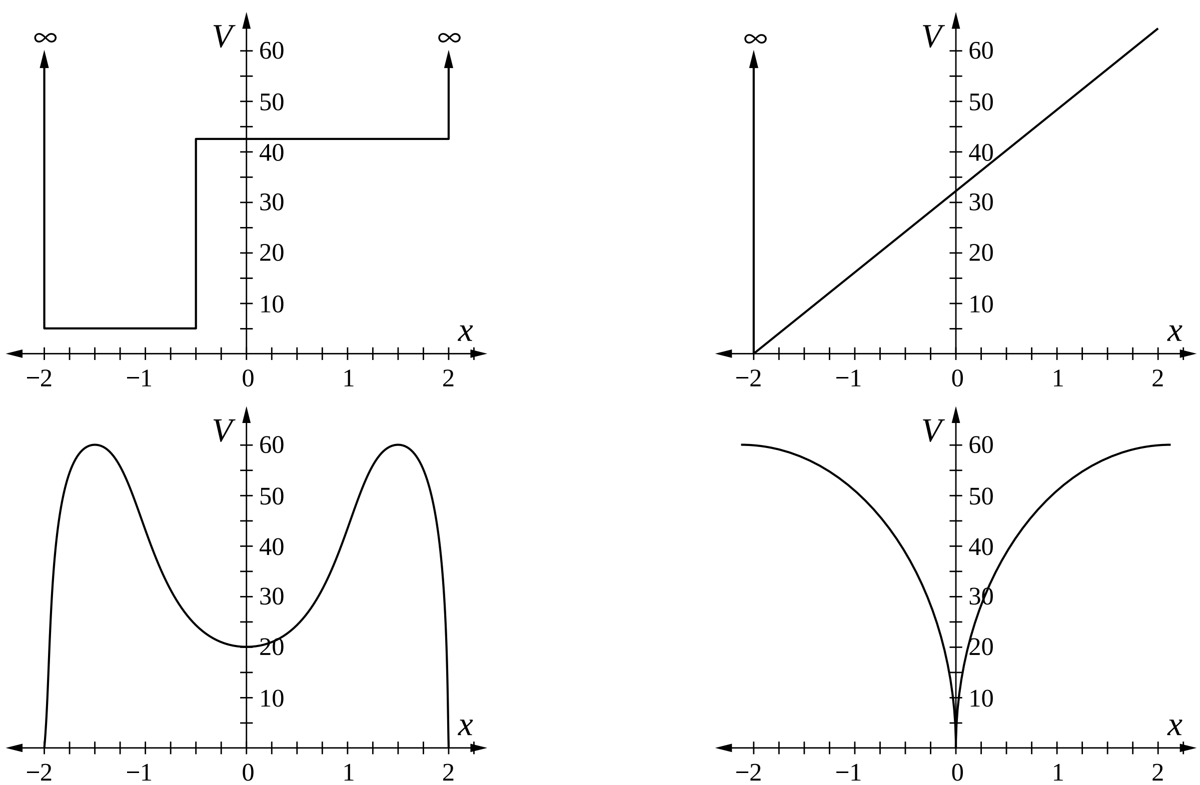
4. (WKB) Kirkman writes the WKB integral in the form:

$$\int_a^b k(x)dx = \pi(n - \text{something}) \qquad \text{where: } k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar} \tag{1}$$

(a) For each of the below four plots of $V(x)$ report the values for a, b, and "something" if we are considering bound state (or quasi bound state) wavefunctions ψ with an energy, E, of 50.

(b) The lower left potential for $E = 50$ has the "quasi bound state" mentioned above. How does this quasi bound state differ from the other states which are truly bound states?

(c) For each of the below potentials, assume that the integral of Eq. 1 produces $n = 10$ for $E = 50$. Sketch the corresponding WKB wavefunction ψ directly on each of the below plots properly displaying changing wavelength & amplitude and behavior near a & b.



5. (WKB) Using the WKB approximation, find the formula for the eigenenergies E of a simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

The following integral may be of use:

$$\int_0^A \sqrt{A^2 - x^2}\, dx = \frac{\pi\, A^2}{4}$$

6. (dTIPT) Consider a particle-in-a-2d-box with $V(x,y) = 0$ for $0 < x < L$ and $0 < y < L$ and $V(x,y) = \infty$ elsewhere. The eigenenergies depend on two whole-number quantum numbers: n_x and n_y:

$$E = \frac{\pi^2 \hbar^2}{2mL^2} \left(n_x^2 + n_y^2 \right)$$

with corresponding normalized eigenfunction:

$$|n_x n_y\rangle = \psi_{n_x n_y}(x,y) = \frac{2}{L} \sin\left(n_x \pi x / L\right) \sin\left(n_y \pi y / L\right)$$

A perturbing 2d delta function potential is placed at the location $(x,y) = (L/3, L/4)$:

$$V' = \lambda\, \delta(x - L/3)\, \delta(y - L/4)$$

Consider the first-order energy shift of the degenerate ψ_{12} and ψ_{21} states. Find the matrix representing V' in this degenerate sub-space. Find the matrix's eigenvalues and vectors. Report the "good" wavefunctions and the corresponding approximate eigenenergy (accurate to first-order) for those "good" wavefunctions. FYI: $\sin(\pi/3) = \sin(2\pi/3) = \sqrt{3}/2$, $\sin(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/2) = 1$