1. Consider the harmonic oscillator (with, as usual, spring constant: $k = m\omega^2$) in one dimension with orthonormal energy eigenfunctions (a.k.a. stationary states) denoted by $\psi_n(x)$ of energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$.

$$\Psi(x) = \frac{1}{\sqrt{6}}\ (\psi_1(x) + 2i\psi_2(x) + \psi_3(x))$$

Calculate $\langle\Psi|p^2|\Psi\rangle$ using raising and lowering operators. What is the probability that an energy of $\frac{5}{2}\,\hbar\omega$ would be measured for this state?

2. Prove the following formulae for commutators. In the below $f(x)$ is an arbitrary function and p is the momentum operator.

   (a) $[p, x]f(x) = \frac{\hbar}{i}\ f(x)$

   (b) $[p, V(x)]f(x) = \frac{\hbar}{i}\ \frac{\partial V}{\partial x}\ f(x)$

   (c) $[p^2, x]f(x) = 2\ \frac{\hbar}{i}\ p\ f(x)$

3. Consider the particle-in-a-box problem with $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ elsewhere with orthonormal energy eigenfunctions denoted $\psi_n(x)$ for $(n = 1, 2, 3, \ldots)$:

$$H\psi_n = E_n\psi_n \qquad \text{where:} \qquad E_n = \frac{(k\hbar)^2}{2m} \qquad \psi_n(x) = \sqrt{\frac{2}{L}}\sin(kx) \qquad k = \frac{n\pi}{L}$$

Normalize the wave function

$$\Psi(x) = \psi_1(x) + i\psi_2(x)$$

Write down the expression for $\Psi(x, t)$ (i.e., a formula for ψ at future times). Directly calculate $\langle\Psi(x, t)|p|\Psi(x, t)\rangle$ and $\langle\Psi(x, t)|x|\Psi(x, t)\rangle$. Show that your answers satisfy the equation:

$$\frac{d}{dt}\ \langle\Psi(x, t)|x|\Psi(x, t)\rangle = \frac{\langle\Psi(x, t)|p|\Psi(x, t)\rangle}{m}$$

The following integrals may be of use:

$$\langle\psi_1|x|\psi_1\rangle = L/2 \qquad \langle\psi_1|p|\psi_1\rangle = 0$$

$$\langle\psi_2|x|\psi_2\rangle = L/2 \qquad \langle\psi_2|p|\psi_2\rangle = 0$$

$$\langle\psi_2|x|\psi_1\rangle = -\frac{16}{9\pi^2}\ L \qquad \langle\psi_2|p|\psi_1\rangle = \frac{8}{3}\ \frac{\hbar}{iL}$$

4. Consider the below operators (matrices):

Operator	Eigen vector	Eigen value	Eigen vector	Eigen value
$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$+1$		-1
$S_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ i \end{pmatrix}$		$\begin{pmatrix} i \\ 1 \end{pmatrix}$	
$S_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$+1$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	-1
$S_{\hat{\mathbf{r}}} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{+i\phi}\sin\theta & -\cos\theta \end{pmatrix}$	$\begin{pmatrix} c \\ e^{+i\phi}s \end{pmatrix}$	$+1$	$\begin{pmatrix} e^{-i\phi}s \\ -c \end{pmatrix}$	-1

For the eigenvectors of $S_{\hat{\mathbf{r}}}$ I used the shorthand: $c \equiv \cos(\theta/2)$ and $s \equiv \sin(\theta/2)$. The parameters θ, ϕ should be considered fixed real constants. Note of no consequence: particular values of θ, ϕ can result in $S_{\hat{\mathbf{r}}}$ being equal to any of the S_i, for example: $\theta = 0, \phi = 0 \Rightarrow S_{\hat{\mathbf{r}}} = S_3$; $\theta = \pi/2, \phi = 0 \Rightarrow S_{\hat{\mathbf{r}}} = S_1$; $\theta = \pi/2, \phi = \pi/2 \Rightarrow S_{\hat{\mathbf{r}}} = S_2$.

(a) Is S_1 Unitary? Is S_2 Hermitian?

(b) Determine and fill in the three missing entries in the above table: 2 eigenvalues and 1 eigenvector.

(c) Show that the two eigenvectors of $S_{\hat{\mathbf{r}}}$ are normalized and orthogonal.

(d) If the particle is known to be in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find the probability that a measurement of $S_{\hat{\mathbf{r}}}$ results in $+1$.

(e) Find $[S_1, S_2]$.