$\int_{-\infty}^{+\infty} e^{-\alpha x^2-\beta x}\,dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha}$ $\quad\int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2}\,dx = \frac{-d}{d\alpha}\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,dx$ $\quad\int_0^{+\infty} x\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$ $\quad\int f(x)\delta(x-a)\,dx = f(a)$

$H\psi = i\hbar\partial_t\psi \qquad H\psi = E\psi \qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\,\partial_x^2 + V(x) \qquad p = -i\hbar\partial_x \qquad [p,x] = -i\hbar$

Particle-in-a-box with $V(x) = 0$ for $0 < x < L$, but $V(x) = \infty$ elsewhere

$E_n = \frac{(\hbar k)^2}{2m} \qquad u_n(x) = \sqrt{\frac{2}{L}}\,\sin(kx)$ where $k = \frac{n\pi}{L} \qquad n = 1,2,3\ldots$

3-d: $|n_x n_y n_z\rangle = u_{n_x}(x)u_{n_y}(y)u_{n_z}(z) \quad E = \frac{(\hbar k)^2}{2m}$ where $\vec{k} = \langle n_x\pi/L_x, n_y\pi/L_y, n_z\pi/L_z\rangle$

Linear Potential with $V(x) = Fx$ for $x > 0$, and $V(x) = \infty$ for $x < 0$ $\qquad$ units: $\ell = \left(\hbar^2/2mF\right)^{\frac{1}{3}} \quad e = \left(\hbar^2F^2/2m\right)^{\frac{1}{3}}$

Delta Function: $V(x) = -\alpha\delta(x) \Longrightarrow \Delta\psi' = -\frac{2m\alpha}{\hbar^2}\,\psi(0) \qquad \psi(x) = \sqrt{\kappa}\,e^{-\kappa|x|}$ where: $\kappa = m\alpha/\hbar^2 \qquad E = -\frac{\kappa^2\hbar^2}{2m}$

Harmonic Oscillator with $V(x) = \frac{1}{2}\,m\omega^2x^2 \qquad E_n = \hbar\omega\,(n+\frac{1}{2}) \qquad n = 0,1,2\ldots$

$|n\rangle = u_n(x) = N_n H_n(\xi)\,e^{-\frac{1}{2}\xi^2}$ where $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ and H_n is an n^{th} degree polynomial

$a_- = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi + \partial_\xi) \qquad a_+ = a_-^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right) = \frac{1}{\sqrt{2}}\,(\xi - \partial_\xi) \qquad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$[a_-, a_+] = 1 \qquad [H, a_\pm] = \pm\hbar\omega a_\pm \qquad H = \hbar\omega\left(\frac{1}{2} + a_+a_-\right) \qquad a_-\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a_+\,|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$

3-d: $\left(n_x + n_y + n_z + \frac{3}{2}\right) = \left(2n_r + \ell + \frac{3}{2}\right) \qquad \psi = \sqrt{\frac{2\cdot n_r!}{(\ell + \frac{1}{2} + n_r)!}}\,r'^{\ell}\,L_{n_r}^{\ell+\frac{1}{2}}(r'^2)\,e^{-r'^2/2}\,Y_{\ell m}(\theta,\phi)$

Angular Momentum: $\vec{L} = \vec{r}\times\vec{p} \qquad [L_i, V_j] = i\hbar\epsilon_{ijk}V_k$ where vector $\vec{V} = \vec{r},\vec{p},\vec{L} \qquad |\ell m\rangle = Y_{\ell m}(\theta,\phi) \qquad -\ell \le m \le +\ell$

$\vec{L}^2\,|\ell m\rangle = \ell(\ell+1)\,\hbar^2\,|\ell m\rangle \qquad L_z\,|\ell m\rangle = m\hbar\,|\ell m\rangle \qquad L_\pm\,|\ell m\rangle = \sqrt{\ell(\ell+1) - m(m\pm 1)}\;\hbar\,|\ell\,m\pm 1\rangle$

$L_\pm = L_x \pm iL_y \qquad [L_+, L_-] = 2\hbar L_z \qquad [L_z, L_\pm] = \pm\hbar L_\pm \qquad [\vec{L}^2, L_\pm] = 0$

Spin $\frac{1}{2}$: $\vec{S} = \frac{\hbar}{2}\,\vec{\sigma} \qquad |\frac{1}{2}\;\frac{1}{2}\rangle = \chi_+ = \uparrow = \begin{pmatrix}1\\0\end{pmatrix} \qquad |\frac{1}{2}\;-\frac{1}{2}\rangle = \chi_- = \downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \sigma_i^2 = 1 \qquad \sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x$

Clebsch-Gordan: $|jm\rangle = \sum C(jm;\ell m_\ell, sm_s)\,|\ell m_\ell\rangle\,|sm_s\rangle$ $\qquad$ know how to use table!

Radial Equation: $\psi(r,\theta,\phi) = Y_{\ell m}(\theta,\phi)R(r) \qquad R(r) = \frac{u(r)}{r}$

$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]R = E\,R \qquad \left[\frac{-\hbar^2}{2m}\partial_r^2 + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]u = E\,u$

H atom: $H = \frac{p^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0 r} \qquad E_n = -\frac{1}{2}\,mc^2\,\frac{(Z\alpha)^2}{n^2} = -\frac{1}{2}\,\frac{Z^2e^2}{4\pi\epsilon_0 a_0 n^2} \approx -13.6\text{ eV}\;\frac{Z^2}{n^2} \qquad n = 1,2,3,\ldots$

$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx .53\text{ Å} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \qquad n = n_r + \ell + 1 \;\therefore\; 0 \le \ell \le n-1 \qquad \rho = \sqrt{\frac{8m|E|}{\hbar^2}}\,r = \frac{2Zr}{na_0}$

$|n\ell m\rangle = R_{n\ell}(\rho)\,Y_{\ell m}(\theta,\phi)$ where $R_{n\ell} = N_{n\ell}\,\rho^\ell\,L_{n_r}^{2\ell+1}(\rho)\,e^{-\frac{1}{2}\rho} \qquad N_{n\ell} = \frac{2}{n^2}\sqrt{\frac{(n-\ell-1)!}{(n+\ell)!}}$

Approximation Methods:

WKB: $\int k(x)\,dx = (n-\frac{1}{2})\pi$ (two linear turning points) $\qquad \hbar k(x) = p(x) = \sqrt{2m(E - V(x))}$

Rayleigh-Ritz: minimize $E = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$

Perturbation Theory: $E_n^1 = \langle n|H'|n\rangle \qquad E_n^2 = \sum_{k\ne n}\frac{|\langle k|H'|n\rangle|^2}{E_n^0 - E_k^0}$ $\qquad$ degenerate: diagonalize matrix $\langle i|H'|j\rangle$

Time Dependent: $c_b(t) \simeq -\frac{i}{\hbar}\int_0^t H'_{ba}(t')\,e^{i(E_b-E_a)t'/\hbar}\,dt'$

if $H' = V(\mathbf{r})\cos\omega t$ then: $P_{a\to b} \approx \frac{|V_{ab}|^2}{\hbar^2}\,\frac{\sin^2\left((\omega_0-\omega)t/2\right)}{(\omega_0-\omega)^2}$ $\qquad$ E1 Unpolarized Light: $R_{a\to b} = \frac{\pi\rho(\omega_0)}{3\epsilon_0\hbar^2}\,|q\langle\psi_b|\vec{\mathbf{r}}|\psi_a\rangle|^2$

Golden Rule: transition rate $= \frac{\pi}{2}\,\frac{|V_{ab}|^2}{\hbar}\times$ density of states

Scattering: $\frac{d\sigma}{d\Omega} = \frac{\text{hits/sec in detector}}{J d\Omega} = \frac{\text{hits/sec in detector}}{nt\text{ beam current }d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = |f|^2$ $\qquad$ where: $\psi \approx e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}$

$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta) \qquad \sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$