Gauss: $\oint \vec{E} \cdot \hat{n}\, da = \dfrac{Q_{enclosed}}{\varepsilon_0}$

$\nabla \cdot E = \rho/\varepsilon_0$ $\qquad$ $\nabla \times E = 0$ $\qquad$ $\nabla^2 \phi = -\dfrac{\rho}{\varepsilon_0}$

Poisson

Laplace

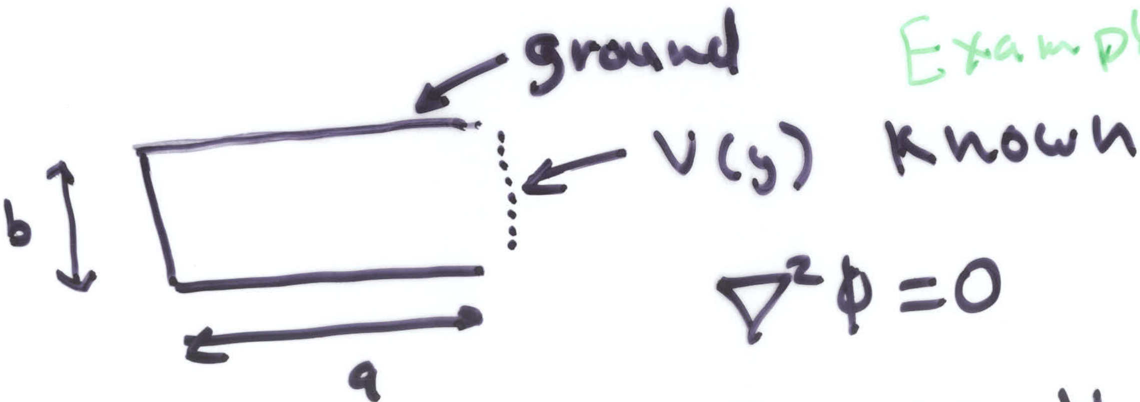
$$\begin{tikzcd} \vec{E} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \phi \\ & \rho \arrow[ur, leftrightarrow] & \end{tikzcd}$$

but ρ hard to "set"

usually control volts

Problem: Given ϕ on boundary find ϕ inside/ouside

Example 5



$$\nabla^2 \phi = 0$$

$$\phi = X(x)\, Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = m^2$$

$X = \begin{smallmatrix}\sinh \\ \cosh\end{smallmatrix} (mx)$

$Y = \begin{smallmatrix}\sin \\ \cos\end{smallmatrix} (my)$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\qquad$ $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$$\sin(\underbrace{mb}_{n\pi}) = V(y=b) = 0 \qquad m = \frac{n\pi}{b}$$

$$\phi_n = \sin\left(\frac{n\pi}{b} y\right) \underbrace{\frac{\sinh\left(\frac{n\pi}{b} x\right)}{\sinh\left(\frac{n\pi}{b} a\right)}}_{1 \quad x=a}$$

$$\phi = \sum A_n \phi_n$$

Find them!

Fourier's Trick

$$\vec{v} = \sum a_i \hat{e}_i = (a_1, a_2, a_3)$$

$$\vec{w} = \sum b_i \hat{e}_i$$

$a(i)$

x

$$\vec{v}\cdot\vec{w} = \sum a_i b_i$$

$$\sum a(x) b(x)\, dx = \int a(x) b(x)\, dx$$

$$\hat{e}_2 \cdot \hat{e}_3 = 0 \qquad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

orthogonal $\perp$

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$

normal

$$\vec{v} = \sum q_i \hat{e}_i$$

$$q_{100} = \vec{v} \cdot \hat{e}_{100}$$

$$q_j = \vec{v} \cdot \hat{e}_j$$

find

$$\phi(x,y) = \sum A_n \phi_n$$

know

$$x = a \qquad \phi(a,y) = V(y)$$

$$\phi_n(\underset{a}{x}, y) = \sin\left(\frac{n\pi}{b} y\right)$$

$$V(y) = \sum A_n \underbrace{\sin\left(\frac{n\pi}{b} y\right)}_{\hat{e}_n}$$

$$\vec{v} =$$

$$\int_0^b \sin\left(\frac{n\pi}{b} y\right) \overbrace{\sin\left(\frac{m\pi}{b} y\right)} dy$$

$$= \delta_{nm} \frac{b}{2}$$

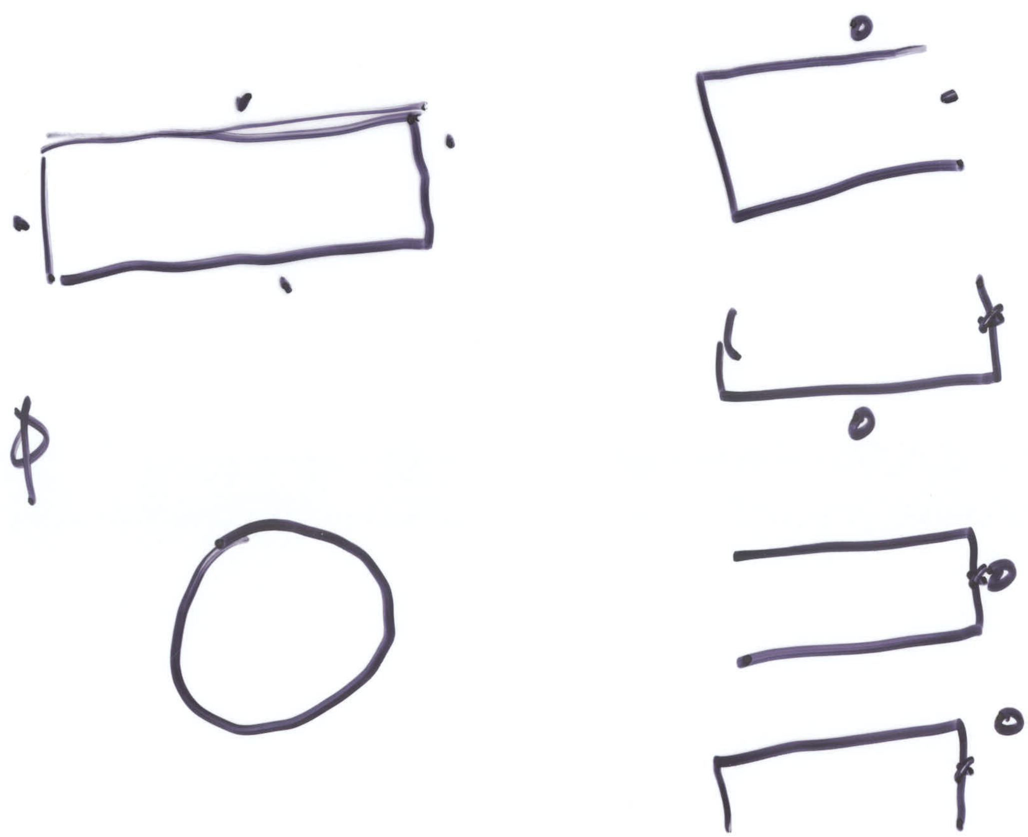
A_m?

$$\int_0^b V(y)\sin\left(\frac{m\pi}{b}y\right)dy \leftarrow \text{Known}$$

$$= \sum A_n \underbrace{\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{b}y\right)}_{\delta_{nm}\frac{b}{2}}$$

$$= A_m \frac{b}{2}$$

$$A_m = \frac{\int_0^b V(y)\sin\left(\frac{m\pi}{b}y\right)dy}{b/2}$$



$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) + \frac{1}{r^2\sin\theta}\left(\partial_\theta \sin\theta\,\partial_\theta\phi\right)$$

$$\quad + \frac{1}{r^2\sin^2\theta}\,\partial_\phi^2\phi$$

$\nabla^2\phi = 0$

$\left.\begin{matrix}\phi\\ \phi\end{matrix}\right\} = 0$

azi symmetric

$$\phi = R(r)\,\Theta(\theta)$$

$$0 = \Theta\left[\frac{1}{\cancel{r^2}}\left(r^2R'\right)'\right]$$

$$+ R\left[\frac{1}{r^2\sin\theta}\left(\sin\theta\,\Theta'\right)'\right]$$

$$0 = \frac{1}{R}\left[\left(r^2R'\right)\right]' + \cancel{\Theta}$$

$$+ \frac{1}{\Theta}\left[\frac{1}{\sin\theta}\left(\sin\theta\,\Theta'\right)'\right]$$

$-l(l+1)$

$$\frac{1}{R}\left[\left(r^2R'\right)\right]' = l(l+1)$$

$$[(r^2R')]' = l(l+1)R$$

$$R = r^q$$

$$R' = q\,r^{q-1}$$

$$r^2R' = q\,r^{q+1}$$

$$(\quad)' = \underline{q(q+1)}\,r^q$$

$$q = l\,,\ -(l+1)$$

$$R = \left(A r^l + \frac{C}{r^{l+1}}\right)$$

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\,\Theta\right) + l(l+1)\,\Theta = 0$$

$$c = \cos\theta \qquad \frac{dc}{d\theta}\frac{d}{dc} = \frac{d}{d\theta}$$

$$-\sin\theta\frac{d}{dc} = \frac{d}{d\theta}$$

$$\frac{d}{dc}\left(\underbrace{(1-c^2)}_{\sin^2\theta}\frac{d}{dc}\,\Theta\right) + l(l+1)\,\Theta = 0$$

$$P_l(c) \leftarrow l^{th} \text{ poly in } c$$

$$P_0 = 1 \quad P_1 = c \quad P_2 = -\frac{1}{2} + \frac{3}{2}c^2$$

$$\int_{-1}^{1} P_n(c)\, P_m(c)\, dc = \delta_{nm} \frac{2}{2n+1}$$

$$\Phi = \sum \left(A_n r^n + \frac{C_n}{r^{n+1}} \right) P_n(c)$$

$V(\theta) : V(c)$

$r = R$

$$\sum A_n r^n P_n(c)$$

$$\Phi(r, \theta) = \sum A_n r^n P_n(c)$$

$$\int_{-1}^{1} x\, P_m(c)$$

even
odd

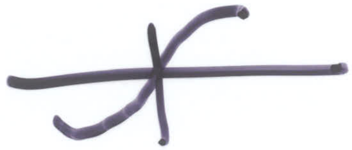
odd

$$\int_{-1}^{1} P_m(c)\, V(c)\, dc = A_m R^m \frac{2}{2m+1}$$

$$\frac{\int_{-1}^{1} P_m V\, dc}{R^m \frac{2}{2m+1}} = A_m$$

$P_0 = 1$ $\quad P_2 = \frac{3}{2}c^2 - \frac{1}{2}$ $\quad P_1 = c$

$$\int_{-L}^{L} \text{odd}\; dx = 0$$



$$\int_{-L}^{L} \text{even}\; dx \neq 0$$

$$= 2\int_{0}^{L} \text{even}\; dx$$

even · even = even

odd · odd = even

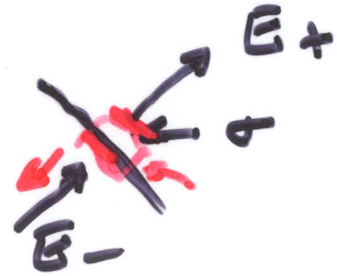
odd · even = odd

$= -E_r$

$\partial_r \phi_{r=R^+}$

$$\partial_r \phi \Big|_{r=R^-}$$

$$= -E_r$$



$$-\nabla \phi = E$$

$$\int E \cdot n\, da = E_{r+} A - E_{r-} A$$

$$= \frac{\sigma A}{\varepsilon_0}$$

$$\underbrace{E_{r+}}_{-\partial_r \phi_+} - \underbrace{E_{r-}}_{\partial_r \phi_-} = \frac{\sigma}{\varepsilon_0}$$